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(6) EVALUATION OF 15TH-ORDER HARMONICS IN THE GEOPOTENTIAL FROM ANALYSIS OF RESONANT ORBITS

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SUMMARY

Satellite orbits contracting under the influence of air drag experience 15th-order resonance when the track over the Earth repeats after 15 revolutions. If the orbital decay rate is slow enough, an orbit passing through the resonance is appreciably perturbed by the effects of 15th-order harmonics in the geopotential. We have used the observed perturbations in 23 resonant orbits, at various inclinations to the equator, to determine the harmonic coefficients of order 15 and degree 15, 16, 17, ... 35. Analysis of the changes in orbital inclination on the 23 orbits gives the harmonics of odd degree, while those of even degree are found from the changes in eccentricity on 16 of the orbits. The values derived are given in Tables 6 and 8. The coefficients of degrees 15, 16, 17, ... 23, should be more accurate than any previously obtained; their average sd is 1.4×10^{-9}, equivalent to 1 cm in geoid height.

Comparisons with comprehensive Earth models showed the Goddard Earth Model 10B to be the best, and a standard deviation of about 3×10^{-9} in the GEM 10B 15th-order coefficients is indicated.

Departmental Reference: Space 591

LIST OF CONTENTS

		Page
1	INTRODUCTION	5
2	THEORY	5
	2.1 General $\beta:\alpha$ resonance	5
	2.2 The 15th-order resonance	7
3	THE ANALYSIS OF THE 23 RESONANT ORBITS	8
	3.1 Pegasus 1, 1965-09A ($i = 31.76^\circ$, $e = 0.007$)	9
	3.2 OSO 6 rocket, 1969-68B ($i = 32.97^\circ$, $e = 0.004$)	11
	3.3 San Marco 1, 1964-84A ($i = 37.80^\circ$, $e = 0.042$)	11
	3.4 HEAO 3, 1979-82A ($i = 43.60^\circ$, $e = 0.001$)	11
	3.5 Tournesol 1 rocket, 1971-30B ($i = 46.36^\circ$, $e = 0.011$)	12
	3.6 Intercosmos 11, 1974-34A ($i = 50.64^\circ$, $e = 0.002$)	14
	3.7 Explorer 44 rocket, 1971-58B ($i = 51.05^\circ$, $e = 0.011$)	14
	3.8 Ariel 1, 1962-15A ($i = 53.82^\circ$, $e = 0.022$)	15
	3.9 Cosmos 72, 1965-53B ($i = 56.04^\circ$, $e = 0.003$)	15
	3.10 Tiros 7 rocket, 1963-24B ($i = 58.20^\circ$, $e = 0.002$)	16
	3.11 Cosmos 373, 1970-87A ($i = 62.92^\circ$, $e = 0.007$)	16
	3.12 Tansei 3 rocket, 1977-12B ($i = 65.49^\circ$, $e = 0.029$)	17
	3.13 Cosmos 462, 1971-106A ($i = 65.70^\circ$, $e = 0.045$)	17
	3.14 China 2 rocket, 1971-18B ($i = 69.84^\circ$, $e = 0.040$)	17
	3.15 Cosmos 387, 1970-111A ($i = 74.00^\circ$, $e = 0.001$)	18
	3.16 Cosmos 395 rocket, 1971-13B ($i = 74.05^\circ$, $e = 0.002$)	18
	3.17 Cosmos 956 rocket, 1977-95B ($i = 75.82^\circ$, $e = 0.029$)	19
	3.18 Ariel 3, 1967-42A ($i = 80.17^\circ$, $e = 0.007$)	19
	3.19 Meteor 3, 1970-19A ($i = 81.16^\circ$, $e = 0.005$)	20
	3.20 OGO 4, 1967-73A ($i = 85.98^\circ$, $e = 0.025$)	21
	3.21 SESP 1, 1971-54A ($i = 90.21^\circ$, $e = 0.002$)	21
	3.22 Nimbus 1 rocket, 1964-52B ($i = 98.68^\circ$, $e = 0.023$)	22
	3.23 OV1-8, 1966-63A ($i = 144.16^\circ$, $e = 0.003$)	22
	3.24 Orbits not used	23
4	THE SOLUTIONS FOR INDIVIDUAL COEFFICIENTS	23
	4.1 The equations to be solved	23
	4.2 The method of solution - a modified least-squares	24
	4.3 The solutions for individual coefficients of odd degree	25
	4.4 The solutions for individual coefficients of even degree	27
5	DISCUSSION	29
6	CONCLUSIONS	31
Acknowledgments		32
Table 1	Values of lumped harmonics $(\bar{C},\bar{S})_{15}^{0,1}$ for the 23 satellites	10
Table 2	Values of lumped harmonics $(\bar{C},\bar{S})_{15}^{1,0}$ and $(\bar{C},\bar{S})_{15}^{-1,2}$ for the 16 satellites	13
Table 3	Values of $Q_{17}^{0,1}$, $Q_{19}^{0,1}$ ... $Q_{39}^{0,1}$ for the 23 satellites	33

LIST OF CONTENTS (concluded)

		Page
Table 4	Values of $Q_{18}^{1,0}$, $Q_{20}^{1,0}$ ... $Q_{38}^{1,0}$ for the 16 satellites	34
Table 5	Values of $Q_{18}^{-1,2}$, $Q_{20}^{-1,2}$ ... $Q_{38}^{-1,2}$ for the 16 satellites	35
Table 6	The values of odd-degree $\bar{C}_{\ell,15}$ and $\bar{S}_{\ell,15}$ given by the 11-coefficient solutions	26
Table 7	Weighted residuals in the 34 equations for odd-degree harmonics, from the 11-coefficient solutions	26
Table 8	The values of even-degree $\bar{C}_{\ell,15}$ and $\bar{S}_{\ell,15}$ given by the 10-coefficient solutions	28
Table 9	Weighted residuals in the 42 equations for even-degree harmonics, from the 10-coefficient solutions	28
Table 10	Comparison of odd-degree 15th-order harmonics up to degree 23, given by GEM 10B and Table 6	30
Table 11	Comparison of even-degree 15th-order harmonics up to degree 24, given by GEM 10B and Table 8	31
References		36
Illustrations		Figures 1-24
Report documentation page		inside back cover

1 INTRODUCTION

The gravitational potential of the Earth is usually expressed as a double infinite series of tesseral harmonics depending on latitude and longitude. The order m of the harmonics expresses the variation with longitude, and a harmonic of order m has m sinusoidal oscillations over 360° of longitude. The degree ℓ of the harmonic (where $\ell > m$) governs variations with latitude; these are more complex and do not concern us here.

If the orbital period of a satellite is such that its successive ground tracks over the Earth are $360^\circ/m$ apart, so that the track repeats after m revolutions, the satellite exhibits mth-order resonance and the perturbations due to harmonics of order m will build up day after day to produce quite a large change in some of the orbital elements. This change can be analysed to determine a lumped harmonic of order m, that is a linear sum of individual harmonics of order m and degree ℓ_0, $\ell_0 + 2$, $\ell_0 + 4$, ..., where $\ell_0 = m$ or $m + 1$ (depending on the orbital element being analysed, and whether m is odd or even). By obtaining values of lumped harmonics for many resonant satellites at different inclinations to the equator, it is possible to solve for the individual harmonics. That is the aim of this paper for harmonics of order $m = 15$, and the values determined here supersede those obtained in Refs 1 and 2.

A satellite experiencing 15th-order resonance has an average height between 470 km (for near-equatorial orbits) and 600 km (for inclination 120°), and at these heights the effects of atmospheric drag are appreciable. So the contraction of the orbit under the influence of air drag brings it to resonance and slowly draws it through resonance. The lower the drag, the longer the resonance acts, and the better the orbit is for analysis. We have analysed 23 orbits: the longest resonance lasts for 5 years; but at some inclinations there are no good specimens and we have to utilise resonances that are effective for only about 2 months.

The theory of the 15th-order resonance is given in section 2. The analyses of the 23 orbits are described in section 3. The results for odd-degree harmonics of order 15, from analysis of inclination, are presented in section 4.3; and results for even-degree harmonics, from analysis of eccentricity, or inclination and eccentricity combined, for 16 of the 23 satellites, are recorded in section 4.4.

2 THEORY

2.1 General $\beta:\alpha$ resonance

The longitude-dependent part of the geopotential at an exterior point (r,θ,λ) can be written in normalized form[3] as

$$\frac{\mu}{r}\sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell}\left(\frac{R}{r}\right)^{\ell} P_{\ell}^{m}(\cos\theta)\left\{\bar{C}_{\ell m}\cos m\lambda + \bar{S}_{\ell m}\sin m\lambda\right\}N_{\ell m}\ , \qquad (1)$$

where r is the distance from the Earth's centre, θ is co-latitude, λ is longitude (positive to the east), μ is the gravitational constant for the Earth ($398600\ km^3/s^2$)

and R is the Earth's equatorial radius (6378.1 km). The $P_\ell^m(\cos\theta)$ are the associated Legendre functions of order m and degree ℓ, and $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$ are the normalized tesseral harmonic coefficients: only those of order $m = 15$ concern us here. The normalizing factor $N_{\ell m}$ is given by[3]

$$N_{\ell m}^2 = \frac{2(2\ell + 1)(\ell - m)!}{(\ell + m)!} \quad . \tag{2}$$

The rate of change of inclination i caused by a relevant pair of geopotential coefficients, $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$, near $\beta:\alpha$ resonance may be written[4,5]

$$\frac{di}{dt} = \frac{n}{\sin i}\left(\frac{R}{a}\right)^\ell \bar{F}_{\ell mp} G_{\ell pq}(k\cos i - m)\mathscr{R}\left[j^{\ell-m+1}(\bar{C}_{\ell m} - j\bar{S}_{\ell m})\exp\{j(\gamma\Phi - q\omega)\}\right] \; , \tag{3}$$

where $\bar{F}_{\ell mp}$ is Allan's normalized inclination function[4], $G_{\ell pq}$ is a function of eccentricity e for which explicit forms have been derived by Gooding[5], $\mathscr{R}$ denotes 'real part of' and $j = \sqrt{-1}$. The resonance angle Φ is defined by the equation

$$\Phi = \alpha(\omega + M) + \beta(\Omega - \nu) \quad , \tag{4}$$

where ω is the argument of perigee, M the mean anomaly, Ω the right ascension of the node and ν the sidereal angle. *The indices* γ, q, k *and* p *in equation (3) are* integers, with γ taking the values 1, 2, 3, ... and q the values 0, ±1, ±2, ...; the equations linking ℓ, m, k and p are: $m = \gamma\beta$; $k = \gamma\alpha - q$; $2p = \ell - k$.

At $\beta:\alpha$ resonance the m-suffix of a relevant $(\bar{C}_{\ell m}, \bar{S}_{\ell m})$ pair is given uniquely by the choice of γ. The values of ℓ to be taken must be such that $\ell \geqslant m$ and $(\ell - k)$ is even. The successive coefficients which arise (for given γ and q) may usefully be gathered together in a lumped form and written as[5]

$$\bar{C}_m^{q,k} = \sum_\ell Q_\ell^{q,k}\bar{C}_{\ell m} \; , \qquad \bar{S}_m^{q,k} = \sum_\ell Q_\ell^{q,k}\bar{S}_{\ell m} \quad , \tag{5}$$

where ℓ increases in steps of 2 from its minimum permissible value ℓ_o, and the $Q_\ell^{q,k}$ are functions of inclination that can be taken as constant for a particular satellite; and $Q_\ell^{q,k} = 1$ when $\ell = \ell_o$.

The rate of change of eccentricity e caused by the (ℓ,m) harmonic near $\beta:\alpha$ resonance can be written[5]

$$\frac{de}{dt} = n(1-e^2)^{-\frac{1}{2}}\left(\frac{R}{a}\right)^\ell \bar{F}_{\ell mp}G_{\ell pq}\left\{\frac{q - \frac{1}{2}(k+q)e^2}{e}\right\}\mathscr{R}\left[j^{\ell-m+1}(\bar{C}_{\ell m} - j\bar{S}_{\ell m})\exp\{j(\gamma\Phi - q\omega)\}\right] \; , \tag{6}$$

with the same definitions as for equation (3).

As the $G_{\ell pq}$ functions are of order $\dfrac{(\frac{1}{2}\ell e)^{|q|}}{(|q|)!}$, it is usually found that, for orbits with eccentricity less than 0.1, the terms with $(\gamma,q) = (1,0)$ produce the most important resonance effects on the inclination, though the terms with $(\gamma,q) = (1,\pm 1)$ also have to be taken into account if e is greater than about 0.03. In the equation for the eccentricity, the relative importance of the terms is largely decided by the value of $\frac{1}{e}G_{\ell pq}\{q - \frac{1}{2}(k+q)e^2\}$, which is of order $\frac{1}{2}ek$ for $q = 0$, of order $\frac{1}{2}\ell$ for $q = \pm 1$, and of order $\frac{1}{2}\ell^2 e$ for $q = \pm 2$. So, for the eccentricity, the strongest effects are usually caused by the terms with $(\gamma,q) = (1,1)$ and $(1,-1)$.

Terms of 30th order, with $\gamma = 2$, may also sometimes need to be taken into account, but their numerical values do not concern us here and will be the subject of a separate paper.

2.2 The 15th-order resonance

For 15th-order resonance ($\beta = 15$, $\alpha = 1$), equation (4) for the resonance angle Φ becomes

$$\Phi = \omega + M + 15(\Omega - \nu) \quad , \tag{7}$$

and at exact resonance $\dot{\Phi} = 0$.

The theoretical equation (3) for variation of inclination may be written as

$$\begin{aligned}\frac{di}{dt} = \frac{n}{\sin i}\left(\frac{R}{a}\right)^{15}\Bigg[&(15 - \cos i)\bar{F}_{15,15,7}\left\{\bar{C}_{15}^{0,1}\sin\Phi - \bar{S}_{15}^{0,1}\cos\Phi\right\}\\ &+ \frac{17e}{2}(15)\left(\frac{R}{a}\right)\bar{F}_{16,15,8}\left\{\bar{S}_{15}^{1,0}\sin(\Phi-\omega) + \bar{C}_{15}^{1,0}\cos(\Phi-\omega)\right\}\\ &+ \frac{13e}{2}(15 - 2\cos i)\left(\frac{R}{a}\right)\bar{F}_{16,15,7}\left\{\bar{S}_{15}^{-1,2}\sin(\Phi+\omega) + \bar{C}_{15}^{-1,2}\cos(\Phi+\omega)\right\}\\ &+ \text{terms in}\left\{\frac{(\frac{1}{2}\ell e)^{|q|}}{(|q|)!}\,{\cos \atop \sin}(\gamma\Phi - q\omega)\right\}\Bigg] \quad ,\end{aligned} \tag{8}$$

where only the three main terms, with $(\gamma,q) = (1,0)$, $(1,1)$ and $(1,-1)$, are given explicitly.

The three pairs of lumped coefficients $\bar{C}_m^{q,k}$ and $\bar{S}_m^{q,k}$ appearing in equation (8) may be written in terms of the individual geopotential coefficients $(\bar{C}_{\ell m},\bar{S}_{\ell m})$ as indicated in equation (5). Explicitly, with the $Q_\ell^{q,k}$ expressed in terms of the $\bar{F}$ functions, the $\bar{C}_m^{q,k}$ are

$$\bar{C}_{15}^{0,1} = \bar{C}_{15,15} - \frac{\bar{F}_{17,15,8}}{\bar{F}_{15,15,7}}\left(\frac{R}{a}\right)^2\bar{C}_{17,15} + \frac{\bar{F}_{19,15,9}}{\bar{F}_{15,15,7}}\left(\frac{R}{a}\right)^4\bar{C}_{19,15} - \ldots \tag{9}$$

$$\bar{C}_{15}^{1,0} = \bar{C}_{16,15} - \frac{19\bar{F}_{18,15,9}}{17\bar{F}_{16,15,8}}\left(\frac{R}{a}\right)^2\bar{C}_{18,15} + \frac{21\bar{F}_{20,15,10}}{17\bar{F}_{16,15,8}}\left(\frac{R}{a}\right)^4\bar{C}_{20,15} - \ldots \tag{10}$$

$$\bar{C}_{15}^{-1,2} = \bar{C}_{16,15} - \frac{15\bar{F}_{18,15,8}}{13\bar{F}_{16,15,7}}\left(\frac{R}{a}\right)^2\bar{C}_{18,15} + \frac{17\bar{F}_{20,15,9}}{13\bar{F}_{16,15,7}}\left(\frac{R}{a}\right)^4\bar{C}_{20,15} - \ldots \tag{11}$$

and similarly for S, on replacing C by S throughout.

For the 15:1 resonance, the theoretical variation of eccentricity given by equation (6) may be written in terms of the same $\bar{C}_m^{q,k}$ and $\bar{S}_m^{q,k}$ as

$$\begin{aligned}\frac{de}{dt} = \frac{n}{2}\left(\frac{R}{a}\right)^{15}\Bigg[\!\!\Bigg[& e\bar{F}_{15,15,7}\left(\bar{C}_{15}^{0,1}\sin\Phi - \bar{S}_{15}^{0,1}\cos\Phi\right) \\ & - 17\left(\frac{R}{a}\right)\bar{F}_{16,15,8}\left\{\bar{S}_{15}^{1,0}\sin(\Phi-\omega) + \bar{C}_{15}^{1,0}\cos(\Phi-\omega)\right\} \\ & + 13\left(\frac{R}{a}\right)\bar{F}_{16,15,7}\left\{\bar{S}_{15}^{-1,2}\sin(\Phi+\omega) + \bar{C}_{15}^{-1,2}\cos(\Phi+\omega)\right\} \\ & + \text{terms in}\left[\frac{(\tfrac{1}{2}\ell)^{|q|}e^{|q|-1}}{\left(|q|\right)!}\left\{q - \tfrac{1}{2}(k+q)e^2\right\}\frac{\cos}{\sin}(\gamma\Phi - q\omega)\right]\Bigg]\!\!\Bigg] .\end{aligned} \tag{12}$$

Three terms are given explicitly in equation (12), those with $(\gamma,q) = (1,0)$, $(1,1)$ and $(1,-1)$. The main terms are expected to be those with $(\gamma,q) = (1,1)$ and $(1,-1)$, but the term with $(\gamma,q) = (1,0)$ is also given, for consistency with equation (8).

3 THE ANALYSIS OF THE 23 RESONANT ORBITS

The methods of analysis have been explained in several previous papers, most recently Refs 6 and 7, and the explanations will not be repeated here. Basically, the observational values of inclination are cleared of irrelevant perturbations and fitted using the computer program THROE[8] with an integrated form of the theoretical equation (8), with extra terms when appropriate, to determine values of the lumped coefficients. Similarly the observational values of eccentricity, cleared of perturbations, are fitted with an integrated form of equation (12), with extra terms as necessary. With a few satellites it is useful to make a simultaneous fitting of inclination and eccentricity using the SIMRES program. In making the fittings, we regard 20 values of inclination (or eccentricity) as the minimum permissible, and we try to analyse orbits over a period of time when $\dot{\Phi}$ lies between −10 and +10 deg/day, though on some high-drag orbits larger values of $\dot{\Phi}$ have to be allowed.

In seeking resonant satellites for analysis, the aim has been to cover the widest possible range of inclination and to leave the smallest possible gaps in the coverage.

No suitable orbits at inclinations less than 30° were found, but this is not so bad as it seems, because such orbits are influenced primarily by harmonics of very high degree (35–55, or even higher for orbits very near the equator) and our evaluations only extend to degree 35. The analyses of the 23 orbits are described in sections 3.1 to 3.23 in order of increasing inclination. The values of lumped harmonics given in sections 3.1 to 3.23 are used (in section 4) to evaluate individual harmonic coefficients. The values of the lumped harmonics are used with their standard deviations unchanged, unless otherwise specified. The values used in the solutions are listed in Tables 1 and 2.

In evaluating the fittings we often refer to the measure of fit, ε , where ε^2 is defined as the sum of squares of weighted residuals divided by the number of degrees of freedom. The weighted residual is the residual of an individual value (of i or e) divided by its assumed standard deviation. For the US Navy orbits used in many of the fittings, the assumed standard deviation is 0.003° in inclination and 0.00004 in eccentricity; for the RAE orbits determined by PROP, the standard deviation given by PROP is used; for other orbits the standard deviation is as specified in the appropriate section.

3.1 Pegasus 1, 1965-09A (i = 31.76°, e = 0.007)

This satellite passed through exact 15th-order resonance on 8 December 1974 and the inclination and eccentricity were analysed over the period September 1974 to May 1975. During this time the rate of change of the resonance angle, $\dot{\Phi}$, increased from -5 to +5 deg/day.

There were 37 US Navy orbits available over the period to be analysed. The values of inclination, cleared of all perturbations except those due to resonance, were first fitted with $(\gamma,q) = (1,0)$ only, the $(1,\pm1)$ terms not being required as the eccentricity was only 0.007. The (1,0) fit was very satisfactory, $\varepsilon = 0.351$, and the values of the lumped harmonics well determined. A second fit was tried with $(\gamma,q) = (1,0)$ and (2,0), but this was not acceptable because the value of ε increased and the $\bar{C}_{30}$ and $\bar{S}_{30}$ coefficients were indeterminate. The $(\gamma,q) = (1,0)$ solution gave the following values:

$$10^9\bar{C}_{15}^{0,1} = 30980 \pm 1960 , \qquad 10^9\bar{S}_{15}^{0,1} = 13540 \pm 960 .$$

The values of inclination and the fitting by THROE are shown in Fig 1.

The values of eccentricity were fitted with $(\gamma,q) = (1,1)$ and $(1,-1)$, as these are the most likely terms to be required in fitting eccentricity alone. The result was very disappointing and none of the terms was determined with sufficient accuracy to be useful, the value of ε being 4.9. A further run with a (0,1) term included was tried. This had been required in previous analyses[6,7] in order to remove an oscillation from the US Navy values of eccentricity, which arises because of a wrong interpretation of the zonal harmonic oscillation removed by the US Navy from their values. However, this technique was not successful here and the values of the lumped coefficients were not acceptable.

Table 1

Values of lumped harmonics $(\bar{C},\bar{S})_{15}^{0,1}$ for the 23 satellites

No.	Satellite	i (deg)	e	$10^9 \bar{C}_{15}^{0,1}$	$10^9 \bar{S}_{15}^{0,1}$	$\bar{F}_{15,15,7}$	$10^9 \bar{F}_{15,15,7} \bar{C}_{15}^{0,1}$	$10^9 \bar{F}_{15,15,7} \bar{S}_{15}^{0,1}$
1	65-09A	31.76	0.007	30980 ± 1960	13540 ± 960	136.3×10^{-6}	4.22 ± 0.27	1.85 ± 0.13
2	69-68B	32.97	0.004	20340 ± 750	6280 ± 910	216.0×10^{-6}	4.39 ± 0.16	1.36 ± 0.20
3	64-84A	37.80	0.042	560 ± 580	-2000 ± 1450	1.111×10^{-3}	0.62 ± 0.64	-2.22 ± 1.61
4	79-82A	43.60	0.001	-467 ± 34	-767 ± 106	5.576×10^{-3}	-2.55 ± 0.19	-4.28 ± 0.59
5	71-30B	46.36	0.011	-596 ± 308†	-869 ± 188†	0.01074	-6.40 ± 3.31	-9.33 ± 2.02
6	74-34A	50.64	0.002	-430.2 ± 10.0	-320.9 ± 8.3	0.02620	-11.27 ± 0.26	-8.41 ± 0.22
7	71-58B	51.05	0.011	-354 ± 94*	-248 ± 45	0.02834	-10.03 ± 2.66	-7.03 ± 1.28
8	62-15A	53.82	0.022	-370 ± 14	-114 ± 31	0.04657	-17.23 ± 0.65	-5.31 ± 1.44
9	65-53B	56.04	0.003	-233.4 ± 3.3	-103 ± 34†	0.06681	-15.59 ± 0.22	-6.88 ± 2.27
10	63-24B	58.20	0.002	-110.6 ± 5.6	-41.6 ± 4.5	0.09207	-10.18 ± 0.52	-3.83 ± 0.41
11	70-87A	62.92	0.007	-5.3 ± 3.2	-32.8 ± 2.5	0.1683	-0.89 ± 0.54	-5.52 ± 0.42
12	77-12B	65.49	0.029	-34 ± 14*	-18 ± 14	0.2217	-7.54 ± 3.10	-3.99 ± 3.10
13	71-106A	65.70	0.045	-36 ± 21	9 ± 17	0.2264	-8.15 ± 4.75	2.04 ± 3.85
14	71-18B	69.84	0.040	-37 ± 6	10 ± 12*	0.3261	-12.07 ± 1.96	3.26 ± 3.91
15	70-111A	74.00	0.001	-26.0 ± 1.0	-5.2 ± 1.3	0.4312	-11.21 ± 0.43	-2.24 ± 0.56
16	71-13B	74.05	0.002	-24.6 ± 1.3	-6.1 ± 1.0	0.4324	-10.63 ± 0.56	-2.64 ± 0.43
17	77-95B	75.82	0.029	-22.5 ± 5.1	-3.0 ± 5.4	0.4745	-10.68 ± 2.42	-1.42 ± 2.56
18	67-42A	80.17	0.007	-23.1 ± 1.6	-8.6 ± 1.3	0.5594	-12.92 ± 0.90	-4.81 ± 0.73
19	70-19A	81.16	0.005	-21.0 ± 1.6	-1.1 ± 5.2†	0.5736	-12.05 ± 0.92	-0.63 ± 2.98
20	67-73A	85.98	0.025	-13.9 ± 2.3	-6.4 ± 3.3	0.6076	-8.45 ± 1.40	-3.89 ± 2.01
21	71-54A	90.21	0.002	-16.40 ± 0.24	-5.37 ± 0.15	0.5855	-9.60 ± 0.14	-3.14 ± 0.09
22	64-52B	98.68	0.023	-28.3 ± 2.0	1.5 ± 8.0†	0.4247	-12.02 ± 0.85	0.64 ± 3.40
23	66-63A	144.16	0.003	72000 ± 16800	-6000 ± 30500†	61.95×10^{-6}	4.46 ± 1.04	-0.37 ± 1.89

Key: * sd ×2 , † sd ×4

3.2 OSO 6 rocket, 1969-68B ($i = 32.97^{\circ}$, $e = 0.004$)

The new fitting of the inclination by THROE, with $(\gamma,q) = (1,0)$ shown in Fig 2, is virtually identical to the previous fitting[2]. The curve fits the points very well, with $\varepsilon = 0.28$, and the values of the lumped harmonics are:

$$10^9\bar{C}^{0,1}_{15} = 20340 \pm 750 \qquad 10^9\bar{S}^{0,1}_{15} = 6280 \pm 910 \quad .$$

A fitting with $(\gamma,q) = (1,0)$ and $(2,0)$ was also tried, but the $(2,0)$ terms were indeterminate. The value of $\dot{\Phi}$ increased from -3.7 deg/day initially to 17.3 deg/day at the end of the analysis.

In analysing the eccentricity, a $(\gamma,q) = (0,1)$ term was added for the reason given in section 3.1. The fitting, with $(\gamma,q) = (1,1)$, $(1,-1)$ and $(0,1)$ terms, was poor, with $\varepsilon = 3.1$. None of the values of lumped harmonics was more than twice its standard deviation. So the attempt was abandoned.

3.3 San Marco 1, 1964-84A ($i = 37.80^{\circ}$, $e = 0.042$)

This is the only 15th-order resonant orbit available at an inclination between 33° and 43°, so it had to be utilized if possible. By an unfortunate chance, the values of both $\bar{C}^{0,1}_{15}$ and $\bar{S}^{0,1}_{15}$ are very small at an inclination of 38°, so the variation in inclination is also very small. To compensate for this ill luck, however, 28 accurate orbits at dates near resonance were available from the archives of the Smithsonian Astrophysical Observatory, and these were analysed[2] in 1974. The last of the 28 orbits does not fit well, and better values for the lumped harmonics have now been obtained by omitting the 28th orbit. The values of inclination, and the fitting by THROE, are shown in Fig 3. The value of $\dot{\Phi}$ runs from -27 to +25 deg/day. Since the eccentricity is appreciable (0.042), we have to use all three (γ,q) terms - $(1,0)$, $(1,1)$ and $(1,-1)$. The assumed accuracy in the values of i was 0.001° and the THROE fitting gave $\varepsilon = 0.50$, with the following values for the lumped coefficients:

$$10^9\bar{C}^{0,1}_{15} = 560 \pm 580 \qquad 10^9\bar{S}^{0,1}_{15} = -2000 \pm 1450 \quad .$$

The eccentricity was also analysed, taking $(\gamma,q) = (1,1)$ and $(1,-1)$. The assumed accuracy was 0.00002 and the value of ε was 2.9. Unfortunately in this fitting and in a simultaneous fitting of i and e with the SIMRES program, the values of lumped harmonics obtained were not accurate enough to be acceptable.

3.4 HEAO 3, 1979-82A ($i = 43.60^{\circ}$, $e = 0.001$)

The third high-energy astronomical observatory was launched on 20 September 1979 into an orbit very close to 15th-order resonance. Exact resonance was reached on 14 November 1979. NASA orbits are available at 2-day intervals and, although it was feared that orbital manoeuvres might have disturbed the effects of the resonance, the analysis proved to be quite satisfactory. Fig 4 gives the values of inclination cleared of perturbations, the fitting by THROE with $(\gamma,q) = (1,0)$ being shown as a broken line. With the accuracy of i taken as 0.002°, the value of ε was 0.63. This fitting follows

81006

the main trends of the variation, but there is obviously an unmodelled oscillation with a period about half that of the argument of perigee ω . So THROE was run with $(\gamma,q) = (1,0)$ and $(0,2)$, that is, with $\sin 2\omega$ and $\cos 2\omega$ terms added. The value of ε decreased to 0.49, the values of the lumped harmonics changed by less than 1 sd, and their standard deviations were reduced by about 20%. This solution, shown by the unbroken line in Fig 4, was preferred, and gives:

$$10^9\bar{C}^{0,1}_{15} = -467 \pm 34 \qquad 10^9\bar{S}^{0,1}_{15} = -767 \pm 106 \; .$$

The values from the $(\gamma,q) = (1,0)$ fitting were -504 ± 41 and -666 ± 132 respectively. These THROE runs used 53 values of i , with $\dot{\Phi}$ increasing from -7.7 to +6.8 deg/day, and gave better results than runs with 55 and 57 values.

The values of eccentricity from the same 53 orbits, cleared of air-drag perturbations, were also successfully fitted with $(\gamma,q) = (1,1)$ and $(1,-1)$, after a residual oscillation correlated with ω was removed by using a value of 0.586 for $10^6 J_3$. The assumed accuracy was 0.00004 and ε was 0.39. Fig 5 shows the values cleared of perturbations and the fitted curve. The values of the lumped harmonics are:

$$10^9\bar{C}^{1,0}_{15} = -860 \pm 150 \qquad 10^9\bar{S}^{1,0}_{15} = -1930 \pm 160$$

$$10^9\bar{C}^{-1,2}_{15} = -234 \pm 34 \qquad 10^9\bar{S}^{-1,2}_{15} = 185 \pm 67 \; .$$

Although the fitting of the curve in Fig 5 is not perfect, only the fourth value needed its standard deviation increased, by a factor of 4, when the values were used in the solutions for individual coefficients.

3.5 Tournesol 1 rocket, 1971-30B ($i = 46.36^\circ$, $e = 0.011$)

Tournesol 1 rocket passed through exact 15th-order resonance on 5 August 1978 and over a period from May to October there were 24 US Navy orbits available for analysis. During this time the value of $\dot{\Phi}$ changed from -9 to +10 deg/day.

The 24 values of inclination, cleared of perturbations except those due to resonance were fitted by THROE with $(\gamma,q) = (1,0)$. This fitting gave $\varepsilon = 0.423$ and the values of the lumped harmonics were:

$$10^9\bar{C}^{0,1}_{15} = -596 \pm 77 \qquad 10^9\bar{S}^{0,1}_{15} = -869 \pm 47 \; .$$

The values of inclination and the THROE fitting with $(1,0)$ are plotted in Fig 6, and although the fit looks good, both standard deviations had to be increased by a factor of 4 in the solutions. Two further runs, with $(\gamma,q) = (1,0)$ and $(2,0)$, and with $(\gamma,q) = (1,0)$, $(1,1)$ and $(1,-1)$, produced larger standard deviations and increased ε .

81006

Table 2

Values of lumped harmonics $(\bar{C},\bar{S})^{1,0}_{15}$ and $(\bar{C},\bar{S})^{-1,2}_{15}$ for the 16 satellites

Satellite	i (deg)	$10^9\bar{C}^{1,0}_{15}$	$10^9\bar{C}^{-1,2}_{15}$	$10^9\bar{S}^{1,0}_{15}$	$10^9\bar{S}^{-1,2}_{15}$	$\bar{F}_{16,15,8}$	$\bar{F}_{16,15,7}$	$10^9\bar{F}_{16,15,8}\bar{C}^{1,0}_{15}$	$10^9\bar{F}_{16,15,7}\bar{C}^{-1,2}_{15}$	$10^9\bar{F}_{16,15,8}\bar{S}^{1,0}_{15}$	$10^9\bar{F}_{16,15,7}\bar{S}^{-1,2}_{15}$
79-82A	43.60	−860 ± 150	−234 ± 34	−1930 ± 160	185 ± 268†	9.277 × 10^{-3}	42.65 × 10^{-3}	−8.0 ± 1.4	−10.0 ± 1.5	−17.9 ± 1.5	7.9 ± 11.4
71-30B	46.36	86 ± 700*	−6 ± 98	−2020 ± 1120†	175 ± 74	18.22 × 10^{-3}	72.35 × 10^{-3}	1.6 ± 12.8	−0.4 ± 7.1	−36.8 ± 20.4	12.7 ± 5.4
74-34A	50.64	−211.2 ± 24.9	−3.0 ± 8.6	128.4 ± 20.2	63.5 ± 3.4	45.17 × 10^{-3}	0.1440	−9.5 ± 1.1	−0.4 ± 1.2	5.8 ± 0.9	9.1 ± 0.5
71-58B	51.05	−466 ± 232†	−50 ± 46*	253 ± 120*	45 ± 14	48.86 × 10^{-3}	0.1526	−22.8 ± 11.3	−7.6 ± 7.0	12.4 ± 5.9	6.9 ± 2.1
62-15A	53.82	−76 ± 18	151 ± 60†	172 ± 76*	11 ± 34	80.16 × 10^{-3}	0.2180	−6.1 ± 1.4	32.9 ± 13.1	13.8 ± 6.1	2.4 ± 7.4
65-53B	56.04	18 ± 17	106.9 ± 8.7	57 ± 23	2.4 ± 8.1	0.1141	0.2780	2.1 ± 1.9	29.7 ± 2.4	6.5 ± 2.6	0.7 ± 2.3
63-24B	58.20	59.3 ± 10.2	101.5 ± 6.5	26.8 ± 7.4	12.9 ± 4.1	0.1551	0.3395	9.2 ± 1.6	34.5 ± 2.2	4.2 ± 1.1	4.4 ± 1.4
71-106A	65.70	51 ± 24	−67 ± 40†	−55 ± 70▽	−19 ± 24	0.3455	0.5129	17.6 ± 8.3	−34.4 ± 20.5	−19.0 ± 24.2	−9.7 ± 12.3
70-111A	74.00	−18.0 ± 3.3	−46.5 ± 2.7	−44 ± 25▽	−40.5 ± 4.0	0.5145	0.4402	−9.3 ± 1.7	−20.5 ± 1.2	−22.6 ± 12.9	−17.8 ± 1.8
71-13B	74.05	−19.8 ± 1.8	−45.5 ± 2.0	−24.8 ± 0.7	−35.2 ± 1.0	0.5149	0.4386	−10.2 ± 0.9	−20.0 ± 0.9	−12.8 ± 0.4	−15.4 ± 0.4
77-95B	75.82	−3 ± 26*	−63 ± 15	−4 ± 22*	−46 ± 18	0.5200	0.3732	−1.6 ± 13.5	−23.5 ± 5.6	−2.1 ± 11.4	−17.2 ± 6.7
67-42A	80.17	−54.7 ± 6.4*	−131 ± 21*	−37.2 ± 2.6	−97 ± 18*	0.4617	0.1552	−25.3 ± 3.0	−20.3 ± 3.3	−17.2 ± 1.2	−15.1 ± 2.8
70-19A	81.16	−26 ± 41▽	−128 ± 29	−15 ± 20†	−130 ± 37	0.4337	0.09806	−11.3 ± 17.8	−12.6 ± 2.8	−6.5 ± 8.7	−12.7 ± 3.6
67-73A	85.98	−87 ± 38	−122 ± 65	84 ± 120*	−175 ± 86	0.2281	−0.1827	−19.8 ± 8.7	22.3 ± 11.9	19.2 ± 27.4	32.0 ± 15.7
71-54A	90.21	−92 ± 48	−62.9 ± 2.6	−170 ± 112*	−53.4 ± 1.6	−0.01237	−0.3833	1.1 ± 0.6	24.1 ± 1.0	2.1 ± 1.4	20.5 ± 0.6
64-52B	98.68	−88 ± 28†	−3 ± 10*	−37 ± 8	−34 ± 11	−0.4287	−0.5139	37.7 ± 12.0	1.5 ± 5.1	15.9 ± 3.4	17.5 ± 5.7

Key: * sd ×2, † sd ×4, ▽ sd ×10

The values of eccentricity were fitted by THROE using $(\gamma,q) = (1,1)$ and $(1,-1)$, and this fitting was accepted as the addition of a (0,1) term gave no advantage. The values of the lumped harmonics were:

$$10^9\bar{C}_{15}^{1,0} = 86 \pm 350 \qquad 10^9\bar{S}_{15}^{1,0} = -2020 \pm 280$$

$$10^9\bar{C}_{15}^{-1,2} = -6 \pm 98 \qquad 10^9\bar{S}_{15}^{-1,2} = 175 \pm 74 \quad .$$

The standard deviations of $\bar{C}_{15}^{1,0}$ and $\bar{S}_{15}^{1,0}$ had to be increased by factors of 2 and 4 respectively. The values of eccentricity and the THROE fitting are plotted in Fig 7.

A SIMRES fitting was tried using $(\gamma,q) = (1,0)$, $(1,1)$ and $(1,-1)$ for both inclination and eccentricity. This gave similar results for eccentricity, but doubled the standard deviations and ε in the fitting of inclination, so the separate fittings were preferred.

3.6 Intercosmos 11, 1974-34A ($i = 50.64^\circ$, $e = 0.002$)

This satellite passed through exact 15th-order resonance on 1 October 1976 and was analysed over a two-year period by Walker[7]. The values for the lumped coefficients from the fitting by THROE of inclination and eccentricity were taken from equations (11) and (16) of Ref 7. They are given in Tables 1 and 2.

3.7 Explorer 44 rocket, 1971-58B ($i = 51.05^\circ$, $e = 0.011$)

The orbit of 1971-58B near 15th-order resonance was determined by Hiller[9] from 700 observations using the RAE computer program PROP. The drag was rather high, so there was no chance of results as good as those from 1974-34A. Mixed PROP and US Navy orbits were used in fitting the inclination, and the analysis was improved by subtracting 0.002° from all the US Navy values of inclination and increasing their standard deviation to 0.005°. Previously, we used the fitting with $(\gamma,q) = (1,0)$, $(2,0)$, $(1,1)$ and $(1,-1)$, but in the light of subsequent experience we regard the (2,0) terms as dubious, and we have chosen the solution with $(\gamma,q) = (1,0)$, $(1,1)$ and $(1,-1)$. The fitting is shown in Fig 8 and the values of the lumped harmonics obtained are:

$$10^9\bar{C}_{15}^{0,1} = -354 \pm 47 \qquad 10^9\bar{S}_{15}^{0,1} = -248 \pm 45 \quad .$$

Since the fitting ($\varepsilon = 1.13$) is not as good as might be hoped, and since accurate values are available from 1974-34A at nearly the same inclination, some relaxation of the standard deviations is reasonable, and that of $\bar{C}_{15}^{0,1}$ was doubled in the final solutions.

Hiller also analysed the eccentricity of this satellite, and his fitting, with $(\gamma,q) = (1,1)$ and $(1,-1)$, is shown in Fig 9. Again, the determination of four coefficients by fitting 28 points is not likely to be reliable, but we decided to use the values and to increase the standard deviation if the residuals in the solutions were poor. The original values were:

81006

$$10^9\bar{C}^{1,0}_{15} = -466 \pm 58 \qquad 10^9\bar{S}^{1,0}_{15} = 253 \pm 60$$

$$10^9\bar{C}^{-1,2}_{15} = -50 \pm 23 \qquad 10^9\bar{S}^{-1,2}_{15} = 45 \pm 14 \ .$$

In the final solutions, the standard deviation was quadrupled for $\bar{C}^{1,0}_{15}$ and doubled for $\bar{S}^{1,0}_{15}$ and $\bar{C}^{-1,2}_{15}$.

3.8 Ariel 1, 1962-15A ($i = 53.82^\circ$, $e = 0.022$)

Ariel 1, the world's first international satellite, passed through exact 15th-order resonance on 8 May 1973. The inclination and eccentricity were analysed by Walker[10] over a period of six months centred on the exact resonance.

Mixed PROP and US Navy orbits were used in the fitting of the inclination and we have used the solution with $(\gamma,q) = (1,0)$ and $(1,1)$. This was the solution recommended by Walker[10] as best and the values of the lumped harmonics are:

$$10^9\bar{C}^{0,1}_{15} = -370 \pm 14 \qquad 10^9\bar{S}^{0,1}_{15} = -114 \pm 31 \ .$$

The values of lumped harmonics obtained from the fitting of eccentricity by THROE were not used in the previous determination of the even harmonic coefficients[1]. Here we use the values from a fitting with $(\gamma,q) = (1,1)$ and $(1,-1)$ given in Run 9 of Table 5 by Walker[10]. The values are:

$$10^9\bar{C}^{1,0}_{15} = -76 \pm 18 \qquad 10^9\bar{S}^{1,0}_{15} = 172 \pm 38$$

$$10^9\bar{C}^{-1,2}_{15} = 151 \pm 15 \qquad 10^9\bar{S}^{-1,2}_{15} = 11 \pm 34 \ .$$

(These values are obtained from the coefficients -B, A, -D and C respectively in Ref 10 after dividing by -0.7602 for the first pair and 1.581 for the second pair.) In the solutions for the individual harmonics it was found necessary to increase the standard deviations of $\bar{C}^{-1,2}_{15}$ and $\bar{S}^{1,0}_{15}$ by factors of 4 and 2 respectively.

3.9 Cosmos 72, 1965-53B ($i = 56.04^\circ$, $e = 0.003$)

This orbit passed slowly through resonance during 1972, with a change in inclination of 0.07°. The previous analysis by THROE, using seven PROP orbits[11] and 45 Navy orbits with $(\gamma,q) = (1,0)$, gave the following values of the lumped harmonics:

$$10^9\bar{C}^{0,1}_{15} = -233.4 \pm 3.3 \qquad 10^9\bar{S}^{0,1}_{15} = -103.4 \pm 8.4 \ .$$

The fit, which is excellent, with $\varepsilon = 0.7$, is shown in Fig 6 of Ref 2. These values fitted well in the previous solutions and the C value was the most accurate previously obtained. However, we found that the S coefficient did not fit our new solutions, which require a value near -60 rather than -103. So further fittings with THROE were tried, including extra terms such as (2,0), (1,1) and (1,-1); omitting the last 6 and then the last 12 values; and then adding 0.0005° to the US Navy values. The values of the

lumped coefficients obtained did not differ significantly from those quoted above, and the numerical value of the S coefficient never fell below 101. So the old values were retained, but the standard deviation of the S coefficient had to be increased by a factor of 4. The non-conforming S value is puzzling, and we plan to determine PROP orbits throughout the resonance phase in the hope of resolving the problem.

In fitting the values of eccentricity, the mismatch between PROP and US Navy values was removed by subtracting $0.0001 \sin \omega$ from the PROP values. The fitting with $(\gamma, q) = (1,1)$ and $(1,-1)$ was fairly satisfactory, with $\varepsilon = 1.62$, but as expected there was an unmodelled variation with the same period as ω. The addition of $(\gamma, q) = (0,1)$ terms led to a much improved fit, with $\varepsilon = 0.80$, shown in Fig 10. The values of the lumped harmonics were as follows:

$$10^9 \bar{C}_{15}^{1,0} = 18 \pm 17 \qquad 10^9 \bar{S}_{15}^{1,0} = 57 \pm 23$$

$$10^9 \bar{C}_{15}^{-1,2} = 106.9 \pm 8.7 \qquad 10^9 \bar{S}_{15}^{-1,2} = 2.4 \pm 8.1 \quad .$$

3.10 Tiros 7 rocket, 1963-24B ($i = 58.20^\circ$, $e = 0.002$)

This satellite, like 1974-34A, was analysed by Walker[7] over a two-year period, exact 15th-order resonance occurring on 3 March 1977. The lumped coefficients from the THROE fittings of inclination and eccentricity are taken from equations (9) and (14) of Ref 7. The values are given in Tables 1 and 2.

3.11 Cosmos 373, 1970-87A ($i = 62.92^\circ$, $e = 0.007$)

The orbit of Cosmos 373 was already past resonance when its initial manoeuvres ceased; apart from this defect, it is a good satellite because its decay rate was slow. Since there are no ideal resonant orbits at inclinations between 58.2° and 74.0°, we have to use four imperfect specimens, of which this is the first.

In 1974 we analysed the variations in inclination using 24 US Navy orbits[2]. The work showed that only $(\gamma, q) = (1,0)$ could be used; the addition of $(1,1)$ and $(1,-1)$ terms was disastrous because of the correlations caused by the near-constancy of ω. A new fitting of the orbits with $(\gamma, q) = (1,0)$ has been made, using improved methods for removing lunisolar and air drag perturbations. The worst-fitting point was the last, so it was omitted. Fig 11 shows the curve fitted by THROE to the 23 orbits, with $\dot{\Phi}$ increasing from 0.7 to 4.6 deg/day. The fitting is very good, with $\varepsilon = 0.31$, and gave the following values for the lumped coefficients:

$$10^9 \bar{C}_{15}^{0,1} = -5.3 \pm 3.2 \qquad 10^9 \bar{S}_{15}^{0,1} = -32.8 \pm 2.5 \quad .$$

The eccentricity was analysed in 1974 but the fitting of the curve was very poor (see Fig 5 of Ref 1), and we decided it was not now acceptable.

The orbit of 1970-87A has been determined by Brookes[12] at selected epochs between 1970 and 1975, and the first five of his epochs fall within the time interval of our analysis. Unfortunately, it was not possible to mix these orbits with the US Navy orbits,

and further orbits are now being determined at the University of Aston in the hope of defining the variations in inclination and eccentricity more precisely.

3.12 <u>Tansei 3 rocket, 1977-12B ($i = 65.49^\circ$, $e = 0.029$)</u>

When the Japanese Tansei 3 satellite was launched on 19 February 1977, a rocket, 1977-12B, was left in a lower orbit. The rocket passed through 15th-order resonance on 16 March 1978 and decayed on 21 March 1979. Because of its high drag, this orbit is far from ideal for resonance analysis.

In the THROE fitting, 25 weekly US Navy orbits were used, covering a range of $\dot{\Phi}$ so wide (-19 deg/day to +29 deg/day) that several oscillations in the 'tail' of the resonance are inevitably included. Fig 12 shows the curve fitted by THROE. As expected, the change at resonance was small, but the fitting was good ($\varepsilon = 0.54$) and the values of the lumped harmonics were:

$$10^9\bar{C}^{0,1}_{15} = -34 \pm 7 \qquad 10^9\bar{S}^{0,1}_{15} = -18 \pm 14 \quad .$$

In the solutions for the individual coefficients, the standard deviation of the first of these values had to be doubled but the second fitted well.

Analysing the variation in eccentricity seemed sure to be fruitless and was not attempted.

The orbit of 1977-12B is now being determined with PROP at the University of Aston and, when this work is completed, more accurate results should be obtained.

3.13 <u>Cosmos 462, 1971-106A ($i = 65.70^\circ$, $e = 0.045$)</u>

The inclination and eccentricity of this high-drag satellite were analysed by Walker[13] and the results from that analysis are used here. The inclination was fitted by THROE with $(\gamma,q) = (1,0)$ and the values obtained were:

$$10^9\bar{C}^{0,1}_{15} = -36 \pm 21 \qquad 10^9\bar{S}^{0,1}_{15} = 9 \pm 17 \quad .$$

The eccentricity values were also analysed and the SIMRES fitting, using $(\gamma,q) = (1,1)$ and $(1,-1)$, was recommended by Walker[13] as the best solution. The values were:

$$10^9\bar{C}^{1,0}_{15} = 51 \pm 24 \qquad 10^9\bar{S}^{1,0}_{15} = -55 \pm 7$$

$$10^9\bar{C}^{-1,2}_{15} = -67 \pm 10 \qquad 10^9\bar{S}^{-1,2}_{15} = -19 \pm 24 \quad .$$

For the $\bar{C}^{-1,2}_{15}$ and $\bar{S}^{1,0}_{15}$ values it was necessary to increase the standard deviations by factors of 4 and 10 respectively.

3.14 <u>China 2 rocket, 1971-18B ($i = 69.84^\circ$, $e = 0.040$)</u>

China 2 rocket was in orbit from 3 March 1971 until 16 February 1976. Its orbit has been determined by Hiller[14] at 114 epochs from more than 7000 observations, using the PROP orbit refinement program. Hiller analysed the orbital changes at four resonances,

and his analysis of inclination at 15th-order resonance with $(\gamma,q) = (1,0)$ gave the following values of lumped harmonics:

$$10^9\bar{C}_{15}^{0,1} = -37 \pm 6 \qquad 10^9\bar{S}_{15}^{0,1} = 10 \pm 6 \quad .$$

The fitting is shown in Fig 11 of Ref 14. Since this was a high-drag orbit, some relaxation of the standard deviations is likely to be necessary; in fact the second was doubled. The values were still most useful, however, since this is our only satellite at inclinations between 65.8° and 74.0°.

Hiller also attempted to analyse the variations in eccentricity, but the resulting values of lumped coefficients were indeterminate[14].

3.15 Cosmos 387, 1970-111A ($i = 74.00^\circ$, $e = 0.001$)

This low-drag satellite gave excellent results[15] from analysis of 19 PROP orbits and 55 US Navy orbits between May 1971 and July 1972. Here we use the same values of the lumped coefficients as before:

$$10^9\bar{C}_{15}^{0,1} = -26.0 \pm 1.0 \qquad 10^9\bar{S}_{15}^{0,1} = -5.2 \pm 1.3$$

$$10^9\bar{C}_{15}^{1,0} = -18.0 \pm 3.3 \qquad 10^9\bar{S}_{15}^{1,0} = -44.1 \pm 2.5$$

$$10^9\bar{C}_{15}^{-1,2} = -46.5 \pm 2.7 \qquad 10^9\bar{S}_{15}^{-1,2} = -40.5 \pm 4.0 \quad .$$

The first two coefficients come from a fitting of inclination with $(\gamma,q) = (1,0)$ and $(2,0)$. The last four coefficients come from a fitting of eccentricity, with $(\gamma,q) = (1,1)$ and $(1,-1)$, being obtained from the coefficients $-B$, A, $-D$ and C respectively in Ref 1, after dividing by -0.6116 for the first pair and by 0.4001 for the second pair. The standard deviation of $\bar{S}_{15}^{1,0}$ had to be increased by a factor of 10; the reason for this discrepancy is not known, but may be connected with the very unusual variation of e resulting from its very low value (see Fig 2 of Ref 15).

3.16 Cosmos 395 rocket, 1971-13B ($i = 74.05^\circ$, $e = 0.002$)

This satellite is almost a twin of 1970-111A and the results, obtained[16] from analysis of 21 PROP orbits and 67 US Navy orbits between September 1971 and October 1972, were even better than for 1970-111A. We used the same values of the lumped coefficients as before:

$$10^9\bar{C}_{15}^{0,1} = -24.6 \pm 1.3 \qquad 10^9\bar{S}_{15}^{0,1} = -6.1 \pm 1.0$$

$$10^9\bar{C}_{15}^{1,0} = -19.8 \pm 1.8 \qquad 10^9\bar{S}_{15}^{1,0} = -24.8 \pm 0.7$$

$$10^9\bar{C}_{15}^{-1,2} = -45.5 \pm 2.0 \qquad 10^9\bar{S}_{15}^{-1,2} = -35.2 \pm 1.0 \quad .$$

The first two coefficients come from a fitting of inclination with $(\gamma,q) = (1,0)$ and $(2,0)$. The last four are from a fitting of eccentricity with $(\gamma,q) = (1,1)$ and $(1,-1)$, 81006

and are obtained from the coefficients -B, A, -D and C respectively in Ref 1, after dividing by -0.6104 for the first pair and by 0.3976 for the second pair.

3.17 Cosmos 956 rocket, 1977-95B ($i = 75.82^{\circ}$, $e = 0.029$)

This is the most recent resonance to be analysed: exact 15th-order resonance occurred on 19 May 1980. Both the inclination and eccentricity have been analysed over a period from mid March until the end of August 1980 using 24 US Navy orbits; during this time $\dot{\Phi}$ changed from -11 to +12 deg/day.

The values of inclination cleared of non-resonant perturbations are plotted in Fig 13 and the curve shows the THROE fitting to the values with $(\gamma,q) = (1,0)$, $(1,1)$ and $(1,-1)$, the $(1,1)$ and $(1,-1)$ terms being included as the eccentricity is 0.029. The lumped coefficients are as follows:

$$10^9\bar{C}^{0,1}_{15} = -22.5 \pm 5.1 \qquad 10^9\bar{S}^{0,1}_{15} = -3.0 \pm 5.4 \ .$$

The values of eccentricity were fitted with THROE over the same period with $(\gamma,q) = (1,1)$ and $(1,-1)$. Adding a $(0,1)$ term was tried for the reason given in section 3.1 but the values were indeterminate. Another fitting with the $(1,0)$ term included was made and used with the THROE fitting of inclination in a SIMRES fitting of inclination and eccentricity together. This procedure proved successful, as the addition of the e-terms from equation (8) helped to provide a better determination of the lumped coefficients, the standard deviations being lower than those obtained from the eccentricity fit alone. The values were:

$$10^9\bar{C}^{1,0}_{15} = -3 \pm 13 \qquad 10^9\bar{S}^{1,0}_{15} = -4 \pm 11$$

$$10^9\bar{C}^{-1,2}_{15} = -63 \pm 15 \qquad 10^9\bar{S}^{-1,2}_{15} = -46 \pm 18 \ .$$

These values were used in the solutions for the individual coefficients, but it was found necessary to double the standard deviations of the two smaller coefficients, $\bar{C}^{1,0}_{15}$ and $\bar{S}^{1,0}_{15}$. The values of eccentricity are plotted in Fig 14 and the curve shows the SIMRES fitting to the values.

The values of the $(1,0)$ terms from the SIMRES fitting were $10^9\bar{C}^{0,1}_{15} = -20.7 \pm 9.2$ and $10^9\bar{S}^{0,1}_{15} = 3.2 \pm 4.5$, with $\varepsilon = 0.960$ as compared with 0.675 from the solution for i alone. These values were not used because of the higher value of ε and also because they did not fit the solutions so well.

3.18 Ariel 3, 1967-42A ($i = 80.17^{\circ}$, $e = 0.007$)

Ariel 3 was the first satellite to be used for evaluating lumped 15th-order harmonics, when Gooding[8] analysed the 281 orbits he had determined from Minitrack observations. Subsequently the orbit has served as a standard for testing in Gooding's development of the THROE and SIMRES programs. The best available values[5] are from a SIMRES fitting of inclination and eccentricity with ten pairs of coefficients, which gives the following values for the lumped harmonic coefficients:

$$10^9\bar{C}_{15}^{0,1} = -23.1 \pm 1.6 \qquad 10^9\bar{S}_{15}^{0,1} = -8.6 \pm 1.3$$

$$10^9\bar{C}_{15}^{1,0} = -54.7 \pm 3.2 \qquad 10^9\bar{S}_{15}^{1,0} = -37.2 \pm 2.6$$

$$10^9\bar{C}_{15}^{-1,2} = -130.6 \pm 10.7 \qquad 10^9\bar{S}_{15}^{-1,2} = -96.8 \pm 9.0 \quad .$$

The $\bar{C}_{15}^{1,0}$, $\bar{C}_{15}^{-1,2}$ and $\bar{S}_{15}^{-1,2}$ coefficients required their standard deviations increased by a factor of 2 in the solutions for individual harmonics.

It is of interest to look back at the values of $10^9\bar{C}_{15}^{0,1}$ and $10^9\bar{S}_{15}^{0,1}$ originally obtained by Gooding 10 years ago[8] with $(\gamma,q) = (1,0)$ only; they were -19.9 ± 1.2 and -7.7 ± 0.8. These are not far from the values required by our final solutions here, namely -22.9 and -8.0.

3.19 Meteor 3, 1970-19A ($i = 81.16^\circ$, $e = 0.005$)

Meteor 3 passed through exact 15th-order resonance on 4 July 1979, and the changes in both inclination and eccentricity have been analysed using 34 US Navy orbits over a period from 11 March to 28 October 1979. During this time $\dot{\Phi}$ changed from -7 to +8 deg/day.

The values of inclination, cleared of perturbations except those due to resonance, were first fitted with THROE using just $(\gamma,q) = (1,0)$. This seemed a fairly satisfactory fit, with $\varepsilon = 0.413$. However, another THROE run with $(\gamma,q) = (1,0)$, $(1,1)$ and $(1,-1)$ was tried, and, although the value of eccentricity was only 0.005, the e-terms in equation (8) were determinate, and ε was reduced to 0.297. The $(1,0)$ terms from this second fit, with lumped coefficients

$$10^9\bar{C}_{15}^{0,1} = -21.0 \pm 1.6 \qquad 10^9\bar{S}_{15}^{0,1} = -1.1 \pm 1.3 \quad ,$$

were accepted, as they gave better results in the solutions for the individual coefficients, although the standard deviation of the $\bar{S}_{15}^{0,1}$ value still had to be increased by a factor of 4. The values of inclination are plotted in Fig 15 and the curve shows the THROE fitting to the values.

The values of eccentricity were fitted with THROE over the same period with $(\gamma,q) = (1,1)$ and $(1,-1)$. The inclusion of $(0,1)$ terms gave no advantage. A further fitting with $(\gamma,q) = (1,0)$, $(1,1)$ and $(1,-1)$, was used with the corresponding THROE fitting of inclination in a SIMRES fit. This yielded values of the lumped coefficients with standard deviations less than those from the THROE fitting with $(\gamma,q) = (1,1)$ and $(1,-1)$. The values were:

$$10^9\bar{C}_{15}^{1,0} = -26.2 \pm 4.1 \qquad 10^9\bar{S}_{15}^{1,0} = -15.0 \pm 5.0$$

$$10^9\bar{C}_{15}^{-1,2} = -128 \pm 29 \qquad 10^9\bar{S}_{15}^{-1,2} = -130 \pm 37 \quad .$$

81006

These values were used in the individual solutions but the standard deviations of $\bar{C}_{15}^{1,0}$ and $\bar{S}_{15}^{1,0}$ had to be increased by factors of 10 and 4 respectively. The values of eccentricity and the curve given by the SIMRES fit are shown in Fig 16.

The values of the $\bar{C}_{15}^{0,1}$ and $\bar{S}_{15}^{0,1}$ lumped coefficients from the SIMRES fitting were not used because they had larger standard deviations than the values from the THROE fitting of inclination alone and they did not fit the solutions so well.

3.20 OGO 4, 1967-73A ($i = 85.98^{\circ}$, $e = 0.025$)

The Orbiting Geophysical Observatory 4 passed through 15th-order resonance on 22 May 1970 and the values of inclination from 27 US Navy orbits were analysed in 1975 [2]. The values obtained then from a THROE fitting with $(\gamma,q) = (1,0)$ and $(1,1)$, the $(1,-1)$ terms being indeterminate, has been accepted, the values being:

$$10^9\bar{C}_{15}^{0,1} = -13.9 \pm 2.3 \qquad 10^9\bar{S}_{15}^{0,1} = -6.4 \pm 3.3 \ .$$

In the previous evaluation[2], the standard deviations had to be relaxed; but here they were used unchanged.

The values of eccentricity were not analysed for the previous determination of even harmonic coefficients[1], but they have now been utilised. A fitting with THROE using $(\gamma,q) = (1,1)$ and $(1,-1)$ was first tried with little success, ε being 3.57. However, when a $(0,1)$ term was included, for the reason given in section 3.1, ε was reduced to 1.47. The values for the lumped coefficients are as follows:

$$10^9\bar{C}_{15}^{1,0} = -87 \pm 38 \qquad 10^9\bar{S}_{15}^{1,0} = 84 \pm 60$$

$$10^9\bar{C}_{15}^{-1,2} = -122 \pm 65 \qquad 10^9\bar{S}_{15}^{-1,2} = -175 \pm 86 \ .$$

These values were used in the solutions for the individual harmonics but it was found necessary to increase the standard deviation of $\bar{S}_{15}^{1,0}$ by a factor of 2. The values of eccentricity and the fitting by THROE are shown in Fig 17.

3.21 SESP 1, 1971-54A ($i = 90.21^{\circ}$, $e = 0.002$)

This orbit passed very slowly through 15th-order resonance between 1972 and 1977, and King-Hele[6] analysed the variations in inclination and eccentricity from 269 weekly US Navy orbits to give the following values of lumped coefficients:

$$10^9\bar{C}_{15}^{0,1} = -16.40 \pm 0.24 \qquad 10^9\bar{S}_{15}^{0,1} = -5.37 \pm 0.15$$

$$10^9\bar{C}_{15}^{1,0} = -92 \pm 48 \qquad 10^9\bar{S}_{15}^{1,0} = -170 \pm 56$$

$$10^9\bar{C}_{15}^{-1,2} = -62.9 \pm 2.6 \qquad 10^9\bar{S}_{15}^{-1,2} = -53.4 \pm 1.6 \ .$$

The first two coefficients came from a fitting with $(\gamma,q) = (1,0)$ and $(2,0)$.

The two eccentricity resonances were four years apart and were analysed separately. The resonance associated with $(\gamma,q) = (1,1)$ was very weak because $\bar{F}_{16,15,8} = 0$ at $i = 90^{\circ}$, and consequently the $(C,S)^{1,0}_{15}$ coefficients are of very poor accuracy: the standard deviation of $\bar{S}^{1,0}_{15}$ had to be doubled in the solutions for individual coefficients. The last two values came from a fitting with $(\gamma,q) = (1,-1)$, $(2,-1)$ and $(1,0)$.

3.22 Nimbus 1 rocket, 1964-52B ($i = 98.68^{\circ}$, $e = 0.023$)

The orbit of Nimbus 1 rocket passed through 15th-order resonance on 5 June 1970, and more than 2000 observations were used by Hiller[17] to determine the orbit with PROP at 25 epochs between March and September 1970. The 25 values of inclination, together with 16 from US Navy orbits, were fitted using THROE with $(\gamma,q) = (1,0)$, $(2,0)$ and $(1,1)$ to give:

$$10^9\bar{C}^{0,1}_{15} = -28.3 \pm 2.0 \qquad 10^9\bar{S}^{0,1}_{15} = 1.5 \pm 2.0 \quad .$$

The variation in inclination, shown in Fig 3 of Ref 17, was surprisingly small. The C value fitted well in the solutions for individual coefficients, but the S value did not, and its standard deviation had to be increased by a factor of 4.

Hiller also fitted the values of eccentricity and inclination together by SIMRES with $(\gamma,q) = (1,0)$, $(2,0)$, $(1,1)$ and $(1,-1)$, and he obtained the following values of lumped harmonics:

$$10^9\bar{C}^{1,0}_{15} = -88 \pm 7 \qquad 10^9\bar{S}^{1,0}_{15} = -37 \pm 8$$

$$10^9\bar{C}^{-1,2}_{15} = -3 \pm 5 \qquad 10^9\bar{S}^{-1,2}_{15} = -34 \pm 11 \quad .$$

(These values are obtained from the coefficients -B, A, -D and C respectively derived in Ref 17, after dividing by 0.5141 for the first pair and -0.4713 for the second pair.) The standard deviations of the C values had to be increased by factors of 4 and 2 respectively when used in the solutions. The fitting of e is shown in Fig 5 of Ref 17.

3.23 OV1-8, 1966-63A ($i = 144.16^{\circ}$, $e = 0.003$)

This satellite passed through exact 15th-order resonance on 19 December 1976. There were 25 US Navy orbits available for analysis from 9 October 1976 to 27 March 1977 and during this period $\dot{\Phi}$ changed from -11 to +14 deg/day.

The 25 values of inclination, cleared of perturbations except those due to resonance, were fitted by THROE using $(\gamma,q) = (1,0)$. This fitting gave $\varepsilon = 0.470$ and the values of the lumped harmonics were:

$$10^9\bar{C}^{0,1}_{15} = 72000 \pm 16800 \qquad 10^9\bar{S}^{0,1}_{15} = -5960 \pm 7630 \quad .$$

The standard deviation on the $\bar{S}^{0,1}_{15}$ value had to be increased by a factor of 4 in the individual solutions. The values of inclination and the curve showing the THROE fit with $(\gamma,q) = (1,0)$ are given in Fig 18.

81006

An analysis of the eccentricity was not attempted as the values were not considered to be accurate enough.

3.24 Orbits not used

Two of the orbits used in our previous determination[2], namely 1964-05A and 1970-65D, have been discarded. The first, at inclination 31.5°, is superseded by 1965-09A, a much better specimen at virtually the same inclination. The second, 1970-65D at 51.2° inclination, was a high-drag orbit included previously only because of a dearth of orbits at inclinations near 50°. Now that 1974-34A is available, there is no need for 1970-65D.

Two new orbits, 1973-99A and 1975-67A, both at inclination 71.0°, were analysed in the hope of improving the coverage at inclinations between 65.7° and 74.0°, where we have only one (high-drag) orbit.

The analysis of 1973-99A utilised 20 US Navy values of inclination between 11 February and 23 June 1974 (resonance was on 29 April); but no clear pattern of variation emerged, and the values of lumped coefficients were indeterminate.

The analysis of 1975-67A was based on 16 US Navy values between 21 September 1975 and 4 January 1976 (resonance being on 7 October 1975). The results were slightly better than for 1973-99A, but the variation in inclination was very feeble and the values of the two lumped harmonics in a $(\gamma,q) = (1,0)$ fitting were:

$$10^9\bar{C}_{15}^{0,1} = -3 \pm 10 \qquad 10^9\bar{S}_{15}^{0,1} = -5 \pm 5 \ ,$$

with $\varepsilon = 0.62$. These values were not considered accurate enough to be worth using, although in retrospect it is apparent that the S value would have fitted well.

We decided not to use the values obtained by Wagner and Klosko[18] from 1971-83B (included in our previous determination[2]), partly because we had already had a good satellite at a similar inclination (33°) and partly because we wished to avoid any direct link with the Goddard Earth Models. In assessing the accuracy of Earth models, it is essential to try to make independent evaluations.

For the same reasons we decided not to use the values obtained from the analyses of inclination at 15th-order resonance by Klokočník[19-21]. These are all at inclinations close to those already represented in our analyses and are most useful as an independent check on our procedures.

4 THE SOLUTIONS FOR INDIVIDUAL COEFFICIENTS

4.1 The equations to be solved

Each of the lumped coefficients derived in sections 3.1 to 3.23 can be expressed as a linear sum of individual 15th-order coefficients, $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$, by equations (9) to (11), which, on reverting to the Q-notation of equation (5), may more compactly be written:

$$\bar{C}^{0,1}_{15} = \bar{C}_{15,15} + Q^{0,1}_{17}\,\bar{C}_{17,15} + Q^{0,1}_{19}\,\bar{C}_{19,15} + \dots \tag{13}$$

$$\left.\begin{aligned} \bar{C}^{1,0}_{15} &= \bar{C}_{16,15} + Q^{1,0}_{18}\,\bar{C}_{18,15} + Q^{1,0}_{20}\,\bar{C}_{20,15} + \dots \\ \bar{C}^{-1,2}_{15} &= \bar{C}_{16,15} + Q^{-1,2}_{18}\,\bar{C}_{18,15} + Q^{-1,2}_{20}\,\bar{C}_{20,15} + \dots \end{aligned}\right\} \tag{14}$$

with similar equations for the S coefficients. The Q constants have been evaluated with the computer program PROF, and all the relevant values for the satellites used are given in Tables 3 to 5 (pages 33 to 35). There are 23 satellites yielding values of the form (13) for odd-degree harmonics, and 16 of them also give equations of the form (14), leading to 32 equations for even-degree harmonics.

Following the procedure which proved successful previously, we add constraint equations of the form

$$\left.\begin{aligned} \bar{C}_{\ell,15} &= 0 \pm 10^{-5}/\ell^2 \\ \bar{S}_{\ell,15} &= 0 \pm 10^{-5}/\ell^2 \end{aligned}\right\} \tag{15}$$

where $\ell = 15, 17, 19 \dots$ for the odd-degree harmonics, and $\ell = 16, 18, 20 \dots$ for those of even degree. These equations express the expectation[3] that the order of magnitude of the individual coefficients of degree ℓ is $10^{-5}/\ell^2$ for $15 < \ell < 50$, as is confirmed in a general way by the Goddard Earth Model 10C (Ref 22).

Thus, when solving for N harmonics, we have 23 + N pairs of equations for odd-degree harmonics, and 32 + N pairs of equations for even-degree harmonics.

4.2 The method of solution - a modified least-squares

Our 23 satellites give results of immensely variable accuracy and reliability. Some, in particular 1974-34A, 1963-24B, 1970-111A, 1971-13B and 1971-54A, are of low drag and give accurate results. Some, such as 1964-84A, 1977-12B, 1971-106A and 1971-18B, are of high drag and are included because they are the only satellites available to fill gaps in the coverage of inclination. Other orbits fall between these extremes.

When the equations were first solved by least squares, it was found, as expected, that some of the values from the less reliable satellites did not fit. Of the 64 values of the coefficients with (q,k) = (1,0) and (-1,2), three appeared to be in error by about 10 sd and another seven by nearly 5 sd.

To attempt a straightforward least-squares solution is an inadequate response to such an abnormal distribution, because large spurious values have too much power in a least-squares fit; or, to put it more ecologically, the normal distribution is the proper habitat for least-squares procedures. One possibility would be to reject at an arbitrary level; but this is a drastic and unsophisticated procedure, a blunt instrument which not only produces discontinuities as the rejection level varies, but also could easily nullify our efforts to provide some representation for unfashionable inclinations. So we rejected all-or-none rejection, and instead introduced two quantum jumps, increasing the

standard deviations of ill-fitting values by factors of either 2 or 4. (We also allowed an increase by a factor of 10, which is closely equivalent to complete rejection and needs no further comment.) This process ensures that, while no ill-fitting value has too much power, the constraint implied by its presence is still operative, though weaker. Also the distribution is brought much closer to normality, thus giving the least-squares process a more natural habitat.

After we had computed and considered a large number of solutions of the equations, it became clear that the optimum number of coefficients would be between 10 and 13 for both odd- and even-degree C and S coefficients. In these circumstances an acceptably distributed set of residuals (with one exception) was obtained by applying the quantum multipliers (2 or 4) so as to keep the weighted residuals from each lumped harmonic less than 1.5. With this choice, the values of the measure of fit ε were all between 0.8 and 1.0, and all but one of the individual C and S coefficients had weighted residuals less than 1.5. Of the 46 odd-degree lumped harmonics, three needed their sd multiplied by a factor of 2, and six needed a factor of 4. Of the 64 even-degree lumped coefficient, twelve needed a factor of 2, seven needed a factor of 4 and three had to be multiplied by 10.

4.3 The solutions for individual coefficients of odd degree

When the 23 equations of type (13) and N equations of type (15) were solved by least squares for N coefficients, the values of the measure of fit ε for $7 \leqslant N \leqslant 13$ were:

N	7	8	9	10	11	12	13
C equations	3.91	2.27	1.15	0.99	0.93	0.92	0.92
S equations	1.21	1.16	1.09	0.84	0.83	0.82	0.82

As before, ε^2 is the sum of squares of weighted residuals divided by the number of degrees of freedom, and the weighted residual is the residual for each lumped coefficient divided by the standard deviation for that coefficient as given in Table 1.

It is obviously advantageous, for both C and S, to solve for at least ten coefficients: it is also necessary, because the Q factors in equation (13) remain quite large up to the 10th for the satellites of lowest inclination, as Table 3 shows. This is confirmed by the behaviour of the solution: in the 9-coefficient S solution the weighted residuals for the first four satellites in Table 1 all numerically exceed 1.0, but in the 10-coefficient solutions they are all less than 0.7.

The solutions for 13 coefficients offer no advantage over those for 12 coefficients, so the choice lies between the 10-, 11- and 12-coefficient solutions. The 11-coefficient solution seems preferable to the 10-coefficient, because of the 6% decrease in ε for the C equations; but the choice between the 11- and 12-coefficient solutions is difficult. The only point in favour of 12 coefficients is that, for one satellite, 1969-68B, the value of the 12th term on the right hand side of (13), $Q_{37}^{0,1} \bar{S}_{37,15}$, exceeds the standard deviation allocated to this satellite, and might therefore seem to be needed - but for the fact that the 11-coefficient solution fits well. The arguments against 12 coefficients are that ε decreases very little and that the 12th coefficient

is formally indeterminate for both C and S. Since only one value (that of $\bar{C}_{27,15}$) changes by more than 0.2 sd on going from 11 to 12 coefficients, the choice is not crucial, and, as it is not crucial, the lower number of coefficients is to be preferred. The 11-coefficient solutions are given in Table 6.

Table 6

The values of odd-degree $\bar{C}_{\ell,15}$ and $\bar{S}_{\ell,15}$ given by the 11-coefficient solutions

ℓ	$10^9\bar{C}_{\ell,15}$	$10^9\bar{S}_{\ell,15}$
15	-22.7 ± 0.6	-7.4 ± 0.6
17	11.3 ± 1.0	6.7 ± 1.2
19	-13.3 ± 0.8	-11.8 ± 0.9
21	15.9 ± 0.7	8.7 ± 0.8
23	14.3 ± 1.6	-1.3 ± 1.9
25	-12.7 ± 2.0	0.6 ± 2.4
27	-6.8 ± 1.4	12.7 ± 2.0
29	-2.2 ± 1.8	0.3 ± 1.9
31	27.9 ± 2.9	-2.0 ± 3.9
33	6.5 ± 2.9	-12.0 ± 3.8
35	-6.8 ± 4.1	3.3 ± 4.6

The weighted residuals in the 23 satellite equations (13) and the 11 constraint equations (15) are given in Table 7. All the residuals of the lumped harmonics are less than 1.5 as a result of applying the quantum multipliers, as explained in section 4.2.

Table 7

Weighted residuals in the 34 equations for odd-degree harmonics, from the 11-coefficient solutions

Satellite equations			Constraint equations		
Satellite	$\bar{C}_{15}^{0,1}$	$\bar{S}_{15}^{0,1}$	Degree ℓ of coefficient	$\bar{C}_{\ell,15}$	$\bar{S}_{\ell,15}$
65-09A	-0.15	0.06	15	0.51	0.17
69-68B	0.02	0.06	17	-0.33	-0.19
64-84A	-0.17	-0.78	19	0.48	0.42
79-82A	0.03	0.38	21	-0.70	-0.38
71-30B	-0.92	-1.01	23	-0.75	0.07
74-34A	0.11	-0.14	25	0.79	-0.04
71-58B	0.85	0.85	27	0.49	-0.93
62-15A	-0.40	0.18	29	0.19	-0.03
65-53B	-0.05	-1.27	31	-2.68	0.19
63-24B	-0.04	0.34	33	-0.70	1.31
70-87A	0.55	-0.33	35	0.83	-0.41
77-12B	-1.17	-0.07			
71-106A	-0.81	1.45			
71-18B	-0.80	0.94			
70-111A	-0.12	0.24			
71-13B	0.91	-0.51			
77-95B	0.31	0.92			
67-42A	-0.27	-0.48			
70-19A	0.48	1.18			
67-73A	0.32	-0.50			
71-54A	-0.07	0.03			
64-52B	-1.00	1.26			
66-63A	1.38	-0.82			

81006

One notable feature of Table 7 is the excellent fitting of nearly all the accurate satellites. The weighted residuals on both the C and S equations are less than 0.3% for 1965-09A, 1969-68B, 1974-34A, 1963-24B, 1970-111A and 1971-54A, and there are no relaxations of the standard deviation on any of these.

The second outstanding feature is the very large value of $\bar{C}_{31,15}$, which is also apparent of course in Table 6. This large value caused us much concern, and we arbitrarily altered the standard deviations of many of the lumped harmonics by a factor of 10 to try to identify the 'culprit' responsible for this high value; but all the satellites indicted had to be acquitted, and the high value seems to result from a consensus. We also computed solutions with the constraints on $\bar{C}_{\ell,15}$ relaxed to $2 \times 10^{-5}/\ell^2$; inevitably the value of $\bar{C}_{31,15}$ increased (from 27.9 to 30.0 $\times 10^{-9}$), the value of ε was reduced (from 0.93 to 0.70), and the standard deviations were also much reduced. This solution was not accepted because it was not compatible with the S solutions. Since making the constraint twice as stringent only reduces this large coefficient from 30.0 to 27.9, we are forced to the conclusion that the value is realistic and is likely to be greater, not less, than 27.9.

The values of the lumped harmonics from 22 of the satellites are plotted against inclination in Fig 19, after multiplication by $\bar{F}_{15,15,7}$ to keep the numerical values to a reasonable level. The satellite omitted is 1966-63A, because its inclination is 144°. The standard deviations in Table 1 are marked as bars. The curves in Fig 19 show the variations given by the 11-coefficient solutions. The fitting is quite satisfactory and it is evident that some of the less accurate values were in need of the increase in standard deviation.

4.4 The solutions for individual coefficients of even degree

When the 32 equations of type (14) and N equations of type (15) were solved by least squares for N coefficients, the values of the measure of fit ε were:

N	7	8	9	10	11	12	13
C equations	1.07	1.06	1.03	0.99	0.99	0.98	0.98
S equations	1.07	0.94	0.93	0.91	0.91	0.91	0.91

The 4% decrease in ε between the 9- and 10-coefficient C solutions is substantial; but there is not much to be said in favour of more than 10 coefficients. The value of the 11th Q-coefficient is most significant with $\bar{C}_{15}^{-1,2}$ for 1971-54A, but the value of $Q_{36}^{-1,2}\bar{C}_{36,15}$ in the 11-coefficient solution is only one-fifth of the standard deviation. So the 11th coefficient should not be required. This is confirmed by the fact that the values of the individual coefficients are not appreciably altered by increasing N beyond 10. So we choose the 10-coefficient solutions, given in Table 8.

Table 8

The values of even-degree $\bar{C}_{\ell,15}$ and $\bar{S}_{\ell,15}$ given by the 10-coefficient solutions

ℓ	$10^9\bar{C}_{\ell,15}$	$10^9\bar{S}_{\ell,15}$
16	-11.0 ± 2.7	-21.5 ± 1.7
18	-43.0 ± 1.8	-22.5 ± 1.2
20	-24.3 ± 2.3	-6.2 ± 1.6
22	24.1 ± 2.0	10.2 ± 1.6
24	1.4 ± 3.8	-21.8 ± 3.3
26	-13.3 ± 5.8	14.4 ± 5.5
28	-15.4 ± 6.4	-8.4 ± 6.3
30	-4.0 ± 6.8	-16.0 ± 6.2
32	7.8 ± 6.2	2.5 ± 5.1
34	9.6 ± 6.3	5.6 ± 5.2

The weighted residuals in the 32 satellite equations (14) and the 10 constraint equations (15) are given in Table 9; all the 84 weighted residuals are less than 1.5.

Table 9

Weighted residuals in the 42 equations for even-degree harmonics, from the 10-coefficient solutions

Satellite equations					Constraint equations		
Satellite	$\bar{C}^{1,0}_{15}$	$\bar{C}^{-1,2}_{15}$	$\bar{S}^{1,0}_{15}$	$\bar{S}^{-1,2}_{15}$	Degree ℓ of coefficient	$\bar{C}_{\ell,15}$	$\bar{S}_{\ell,15}$
79-82A	-0.08	-0.13	-0.30	0.85	16	0.28	0.55
71-30B	0.80	1.07	-1.33	0.59	18	1.40	0.73
74-34A	-0.26	-0.68	-0.23	0.01	20	0.97	0.25
71-58B	-1.19	-1.38	0.94	-0.64	22	-1.16	-0.49
62-15A	0.80	1.24	0.91	-0.12	24	-0.08	1.25
65-53B	1.33	0.45	0.26	-0.88	26	0.90	-0.97
63-24B	-0.97	0.66	0.24	0.53	28	1.20	0.65
71-106A	-0.80	-1.24	-0.95	0.12	30	0.36	1.44
70-111A	0.25	-0.23	-0.77	-1.27	32	-0.80	-0.26
71-13B	-0.46	0.34	0.01	0.36	34	-1.10	0.65
77-95B	0.82	-0.26	1.01	-0.20			
67-42A	-1.21	-1.38	-0.40	-0.96			
70-19A	0.67	-0.29	1.27	-0.82			
67-73A	-0.28	-0.85	1.23	-1.41			
71-54A	-0.23	0.60	-0.87	0.20;			
64-52B	-1.22	1.38	0.50	-1.14			

The values of $\bar{F}_{16,15,8}\bar{C}^{1,0}_{15}$ and $\bar{F}_{16,15,7}\bar{C}^{-1,2}_{15}$ are plotted against inclination in Fig 20, and Fig 21 is a similar diagram for the S coefficients. It should be remembered that the two sets of values in Fig 20 are being fitted simultaneously: thus the rather perverse-looking course of the curve for $\bar{C}^{1,0}_{15}$ in Fig 20 near the $i = 65.7^{\circ}$ point is caused by the need to fit one of the $\bar{C}^{-1,2}_{15}$ values ($i = 58.2^{\circ}$) which has similar coefficients. If we make allowance for this effect, the fitting in Figs 20 and

21 is entirely satisfactory, $\bar{S}_{15}^{-1,2}$ being particularly good. It is also apparent that a few of the values thoroughly deserve their increased standard deviation.

5 DISCUSSION

In comparing our new values for the 15th-order coefficients with previous values, there is a problem in deciding which previous sets of coefficients should be considered. Klokočník and Pospíšilová[23] have pointed out that the variation of lumped coefficients with inclination provides a useful test of the accuracy of existing geopotential models, because the values predicted by the models can be compared with each other and also with the more accurate values derived independently by resonance analysis for particular inclinations. Klokočník and Pospíšilová present the variations of selected lumped coefficients, including $\bar{F}_{15,15,7}(\bar{C},\bar{S})_{15}^{0,1}$, as given by 11 different geopotential models derived in the past 10 years. Of these 11 models, four can be excluded because they do not go beyond degree 18. Of the remaining seven, three are Goddard Earth Models, of which the latest, GEM 10B [24], can be taken as superseding the earlier ones. (We do not require GEM 10C [22], which is the same as GEM 10B to degree 36, because we do not go beyond degree 36 in our solutions.) We also exclude HARMOGRAV, which appears[23] to be the least accurate of the remaining models.

This process of elimination leaves us with four models, namely GEM 10B; the Smithsonian Standard Earth IV.3 (Gaposchkin[25], labelled as SE '5' by Klokočník and Pospíšilová); GRIM 2 (the European model[26]); and the purely terrestrial model 'Rapp 1977' (Ref 27). The variations of $\bar{F}_{15,15,7}\bar{C}_{15}^{0,1}$ and $\bar{F}_{15,15,7}\bar{S}_{15}^{0,1}$ for these four models, as given by Klokočník and Pospíšilová, are reproduced in Figs 22 and 23. Our values of $10^9\overline{FC}$ and $10^9\overline{FS}$ from Table 1 are also plotted whenever their standard deviation is less than 1.0, and are marked as black circles of diameter 1.0. Fig 23 shows that the values of $\overline{FS}$ given by SSE IV.3 and Rapp 1977 cannot be regarded as realistic, because they differ so greatly from those experienced by the resonant orbits for inclinations between 70° and 90°. At 90°, for example, 1971-54A gives $10^9\overline{FS} = -3.1 \pm 0.1$, while SSE IV.3 gives +24 and Rapp 1977 gives +18.

So we are reduced to GEM 10B and GRIM 2 for our comparisons.. Since GRIM 2 utilized our previous values for 15th-order harmonics, almost unchanged, it does not provide an independent test. But it does show approximately how a curve based on our old values of $\bar{C}_{\ell,15}$ and $\bar{S}_{\ell,15}$ fits the new data. Looking first at Fig 23 (which is the worse), we see that the first two points conform with the dot-dash curve, but the three new points at inclinations of 43.6°, 50.6° and 58.2° do not; there are smaller differences for some of the 80°-90° points where we have used new values. In Fig 22 the situation is similar, except that the 58° point fits well. All this is as expected and merely confirms the truism that new data demand a new solution.

The only useful comparison therefore is with GEM 10B, which is believed to be independent of our results. Figs 22 and 23 show that GEM 10B agrees quite well with the values from the resonant orbits and can therefore be accepted as a fairly good representation of the 15th-order harmonics of odd degree. The individual odd degree values in GEM 10B differ considerably from ours, however, for degree 25 and higher, and only time

will tell which is the better set of individual coefficients; the average standard deviation of our odd-degree coefficients for degrees 25 to 35 is 2.8×10^{-9}, while the standard deviation of the GEM 10B values is estimated[6,7] as 3×10^{-9}. For the odd-degree harmonics of degree 15 to 23, the values of the coefficients can be regarded as quite well established, and Table 10 compares the values from GEM 10B with our solution.

Table 10

Comparison of odd-degree 15th-order harmonics up to degree 23, given by GEM 10B and Table 6

ℓ	$10^9\bar{C}_{\ell,15}$		$10^9\bar{S}_{\ell,15}$	
	GEM 10B	Table 6	GEM 10B	Table 6
15	-19.7	-22.7 ± 0.6	-6.4	-7.4 ± 0.6
17	2.5	11.3 ± 1.0	4.8	6.7 ± 1.2
19	-20.6	-13.3 ± 0.8	-15.3	-11.8 ± 0.9
21	15.2	15.9 ± 0.7	9.5	8.7 ± 0.8
23	15.4	14.3 ± 1.6	4.1	-1.3 ± 1.9

If the GEM 10B standard deviation is taken as 3×10^{-9} for all values, the difference between GEM 10B and the corresponding value in our solution is, on average, 0.8 × (the sum of the two standard deviations), the S coefficients being in better agreement than the C coefficients. If the sets of values are independent, as we believe, the agreement is very satisfactory.

The variations of the even-degree GEM 10B lumped harmonics $(\bar{C},\bar{S})_{15}^{1,0}$ and $(\bar{C},\bar{S})_{15}^{-1,2}$, multiplied by the appropriate $\bar{\bar{F}}$ factors, are shown in Fig 24. Also plotted are those values of lumped harmonics determined from the resonant orbits (Table 2) which have standard deviations less than 2.5. The points are plotted as circles of diameter 2.5. Again the impression is that GEM 10B provides quite a good approximation to the observed values, the worst discrepancy being for $\bar{S}_{15}^{1,0}$ at 44° inclination.

The comparison of the first five individual coefficients, Table 11, shows excellent agreement for the C but greater discrepancies for the S coefficients. If the standard deviation of the GEM 10B values is taken as 3.0, the difference between GEM 10B and the corresponding value in our solution is, on average, 1.2 × (the sum of the two standard deviations) which is rather high but acceptable. In view of the good fitting of the GEM 10B curve to our lumped S values in Fig 24, it may seem surprising that there are such large differences between the GEM 10B $\bar{S}_{\ell,15}$ values and ours. But this is to be expected because our fitted curves (Fig 21) differ considerably from the GEM 10B curves.

Table 11

Comparison of even-degree 15th-order harmonics up to degree 24, given by GEM 10B and Table 8

ℓ	$10^9\bar{C}_{\ell,15}$		$10^9\bar{S}_{\ell,15}$	
	GEM 10B	Table 8	GEM 10B	Table 8
16	-14.4	-11.0 ± 2.7	-27.8	-21.5 ± 1.7
18	-48.3	-43.0 ± 1.8	-18.6	-22.5 ± 1.2
20	-23.9	-24.3 ± 2.3	4.8	-6.2 ± 1.6
22	24.1	24.1 ± 2.0	-1.3	10.2 ± 1.6
24	3.1	1.4 ± 3.8	-5.1	-21.8 ± 3.3

Our new values of the individual coefficients are formally much more accurate than our previous values[1,2] (the standard deviation being halved, on average); the new values should also be more reliable because the coverage of inclination is much better.

6 CONCLUSIONS

The aim of this paper has been to derive the best possible values of 15th-order harmonic coefficients of odd and even degree, solely from analysis of orbits which have passed through 15th-order resonance. We have tried to obtain as many reliable values as possible of lumped 15th-order harmonics from orbits over a wide range of inclinations. Useful values of lumped harmonics of *odd* degree were determined from analysis of the changes in inclination of 22 orbits at inclinations between 30° and 100°, and one at 144°. The distribution in inclination was far more satisfactory than has been achieved before, but there are still some gaps that need filling. Useful values of lumped harmonics of *even* degree were obtained by analysing the changes in orbital eccentricity of 16 of the 23 orbits, and this yielded 32 pairs of equations for coefficients of even degree.

These values of lumped harmonics have been used to solve for individual coefficients, and the solutions chosen as the best, given in Tables 6 and 8, are the 11-coefficient solution for odd degree and the 10-coefficient for even degree, so that the solution is complete for degree 15-35. The coefficients of degree 15-23 should be more accurate than any obtained previously, but the errors inevitably increase as the degree increases beyond 24.

The values of lumped harmonics also provide an independent test of comprehensive Earth models. Figs 22 and 23 suggest that only the Goddard Earth Model 10B is useful for comparison. Tables 10 and 11 compare our values of 15th-order coefficients with those of GEM 10B for degree up to 24. If it is assumed that the GEM 10B coefficients have standard deviations of 3×10^{-9}, the difference between the twenty GEM 10B values and the corresponding values in our solution is, on average, 1.0 × (the sum of the two standard deviations). If the sets of values are independent, as we believe, this agreement is very satisfactory.

The average standard deviation of our coefficients of degree 15, 16, 17 .... 23 is 1.4×10^{-9}, equivalent to 1 cm in geoid height.

Acknowledgments

We are most grateful to R.H. Gooding, whose work in developing theory and computer programs for the analysis of resonance has been essential to our endeavours. We also thank H. Hiller, who analysed three of the orbits utilized in the evaluation, and we are indebted to J. Klokočník for making available his results on the intercomparison of Earth models.

Table 3

VALUES OF $Q_{17}^{0,1}$, $Q_{19}^{0,1}$ ... $Q_{39}^{0,1}$ FOR THE 23 SATELLITES

Satellite	$Q_{17}^{0,1}$	$Q_{19}^{0,1}$	$Q_{21}^{0,1}$	$Q_{23}^{0,1}$	$Q_{25}^{0,1}$	$Q_{27}^{0,1}$	$Q_{29}^{0,1}$	$Q_{31}^{0,1}$	$Q_{33}^{0,1}$	$Q_{35}^{0,1}$	$Q_{37}^{0,1}$	$Q_{39}^{0,1}$
65-09A	-12	59	-186	426	-755	1058	-1166	963	-490	-48	397	-416
69-68B	-11.5	55.1	-166.3	362.0	-600.6	771.1	-750.2	497.8	-113.0	-208.8	309.9	-183.2
64-84A	-10.0	40.6	-99.8	167.1	-194.4	143.8	-34.3	-61.6	79.6	-25.2	-35.0	43.7
79-82A	-8.21	25.64	-44.67	44.93	-18.22	-13.86	21.25	-2.92	-12.92	7.67	5.90	-7.42
71-30B	-7.32	19.59	-27.19	17.57	3.13	-12.98	3.87	7.22	-4.74	-3.77	4.01	1.94
74-34A	-5.97	11.94	-9.87	-0.53	6.22	-1.10	-3.92	0.91	2.65	-0.46	-1.86	0.09
71-58B	-5.84	11.30	-8.71	-1.24	5.74	-0.34	-3.70	0.29	2.50	0.02	-1.71	-0.27
62-15A	-5.003	7.539	-2.913	-3.325	2.152	2.155	-1.147	-1.639	0.384	1.218	0.110	-0.802
65-53B	-4.351	5.103	-0.257	-2.867	-0.045	1.804	0.533	-1.012	-0.748	0.370	0.677	0.066
63-24B	-3.740	3.202	1.066	-1.745	-1.071	0.721	1.005	0.013	-0.665	-0.381	0.231	0.401
70-87A	-2.509	0.496	1.350	0.340	-0.615	-0.634	-0.073	0.361	0.343	0.045	-0.194	-0.201
77-12B	-1.905	-0.278	0.819	0.693	0.039	-0.403	-0.385	-0.094	0.166	0.226	0.113	-0.040
71-106A	-1.858	-0.323	0.770	0.697	0.082	-0.368	-0.386	-0.123	0.138	0.219	0.127	-0.020
71-18B	-1.014	-0.759	-0.093	0.341	0.409	0.231	-0.003	-0.155	-0.180	-0.110	-0.011	0.062
70-111A	-0.331	-0.587	-0.454	-0.200	0.027	0.160	0.192	0.151	0.075	0.002	-0.049	-0.068
71-13B	-0.323	-0.583	-0.455	-0.204	0.021	0.156	0.190	0.152	0.078	0.005	-0.046	-0.067
77-95B	-0.087	-0.411	-0.429	-0.307	-0.144	-0.002	0.090	0.126	0.119	0.084	0.040	0.000
67-42A	0.346	0.059	-0.097	-0.171	-0.192	-0.177	-0.144	-0.102	-0.061	-0.026	0.002	0.020
70-19A	0.416	0.153	-0.003	-0.092	-0.136	-0.147	-0.138	-0.116	-0.089	-0.060	-0.035	-0.013
67-73A	0.588	0.420	0.313	0.236	0.178	0.132	0.097	0.069	0.047	0.031	0.018	0.009
71-54A	0.513	0.327	0.221	0.154	0.110	0.079	0.058	0.042	0.031	0.023	0.017	0.012
64-52B	-0.268	-0.515	-0.466	-0.324	-0.175	-0.054	0.029	0.076	0.095	0.093	0.079	0.059
66-63A	-14	77	-266	651	-1198	1693	-1824	1398	-559	-254	613	-435

Table 4

VALUES OF $Q_{18}^{1,0}$, $Q_{20}^{1,0}$ ... $Q_{38}^{1,0}$ FOR THE 16 SATELLITES

Satellite	$Q_{18}^{1,0}$	$Q_{20}^{1,0}$	$Q_{22}^{1,0}$	$Q_{24}^{1,0}$	$Q_{26}^{1,0}$	$Q_{28}^{1,0}$	$Q_{30}^{1,0}$	$Q_{32}^{1,0}$	$Q_{34}^{1,0}$	$Q_{36}^{1,0}$	$Q_{38}^{1,0}$
79-82A	-6.2	18.9	-35.0	39.2	-20.0	-10.4	23.4	-7.7	-13.4	12.8	4.4
71-30B	-5.46	14.35	-21.14	15.44	1.54	-12.56	5.77	6.96	-6.96	-3.20	6.16
74-34A	-4.42	8.59	-7.43	-0.39	5.59	-1.48	-3.92	1.54	2.93	-1.13	-2.30
71-58B	-4.32	8.11	-6.51	-1.02	5.19	-0.70	-3.79	0.81	2.88	-0.49	-2.24
62-15A	-3.67	5.27	-1.95	-2.84	1.97	2.01	-1.35	-1.71	0.69	1.48	-0.11
65-53B	-3.16	3.44	0.10	-2.42	-0.08	1.78	0.48	-1.19	-0.79	0.59	0.86
63-24B	-2.67	2.02	1.07	-1.40	-1.04	0.71	1.06	-0.04	-0.82	-0.42	0.38
71-106A	-1.160	-0.534	0.506	0.698	0.180	-0.357	-0.461	-0.174	0.174	0.308	0.185
70-111A	0.134	-0.404	-0.525	-0.358	-0.085	0.148	0.264	0.254	0.157	0.030	-0.075
71-13B	0.141	-0.399	-0.524	-0.362	-0.092	0.142	0.261	0.255	0.161	0.036	-0.070
77-95B	0.355	-0.170	-0.428	-0.442	-0.300	-0.101	0.077	0.187	0.216	0.178	0.102
67-42A	0.780	0.457	0.152	-0.084	-0.237	-0.308	-0.312	-0.267	-0.195	-0.113	-0.036
70-19A	0.857	0.594	0.320	0.081	-0.100	-0.217	-0.275	-0.282	-0.254	-0.202	-0.140
67-73A	1.112	1.105	1.041	0.946	0.834	0.716	0.599	0.487	0.385	0.294	0.214
71-54A	1.177	1.250	1.270	1.256	1.220	1.169	1.108	1.040	0.970	0.899	0.828
64-52B	0.861	0.605	0.339	0.106	-0.071	-0.187	-0.247	-0.260	-0.239	-0.195	-0.140

Table 5

VALUES OF $Q_{18}^{-1,2}$, $Q_{20}^{-1,2}$ ... $Q_{38}^{-1,2}$ FOR THE 16 SATELLITES

Satellite	$Q_{18}^{-1,2}$	$Q_{20}^{-1,2}$	$Q_{22}^{-1,2}$	$Q_{24}^{-1,2}$	$Q_{26}^{-1,2}$	$Q_{28}^{-1,2}$	$Q_{30}^{-1,2}$	$Q_{32}^{-1,2}$	$Q_{34}^{-1,2}$	$Q_{36}^{-1,2}$	$Q_{38}^{-1,2}$
79-82A	-4.37	9.12	-10.09	3.79	4.61	-5.97	-0.29	4.80	-1.76	-3.11	2.41
71-30B	-3.76	6.27	-4.42	-1.23	4.05	-0.76	-2.89	1.34	1.98	-1.37	-1.36
74-34A	-2.85	2.89	0.13	-2.18	0.28	1.66	-0.19	-1.30	0.00	1.01	0.17
71-58B	-2.76	2.63	0.35	-2.04	0.01	1.57	0.07	-1.21	-0.22	0.90	0.36
62-15A	-2.19	1.15	1.18	-0.89	-0.99	0.49	0.89	-0.11	-0.73	-0.19	0.49
65-53B	-1.75	0.29	1.19	-0.02	-0.91	-0.32	0.57	0.52	-0.18	-0.49	-0.13
63-24B	-1.34	-0.27	0.88	0.51	-0.43	-0.62	-0.03	0.46	0.32	-0.14	-0.35
71-106A	-0.104	-0.644	-0.467	0.031	0.384	0.378	0.109	-0.173	-0.274	-0.170	0.022
70-111A	0.874	0.513	0.096	-0.245	-0.431	-0.446	-0.327	-0.140	0.044	0.173	0.223
71-13B	0.879	0.521	0.106	-0.237	-0.427	-0.448	-0.333	-0.149	0.036	0.168	0.221
77-95B	1.032	0.820	0.481	0.117	-0.188	-0.381	-0.443	-0.391	-0.263	-0.105	0.040
67-42A	1.407	1.634	1.687	1.581	1.351	1.038	0.689	0.347	0.045	-0.192	-0.353
70-19A	1.59	2.02	2.26	2.31	2.18	1.92	1.56	1.14	0.73	0.34	0.01
67-73A	0.988	0.879	0.741	0.598	0.461	0.334	0.222	0.125	0.044	-0.023	-0.075
71-54A	0.957	0.839	0.714	0.598	0.496	0.410	0.337	0.276	0.226	0.185	0.151
64-52B	0.236	-0.261	-0.468	-0.469	-0.357	-0.203	-0.054	0.063	0.138	0.172	0.172

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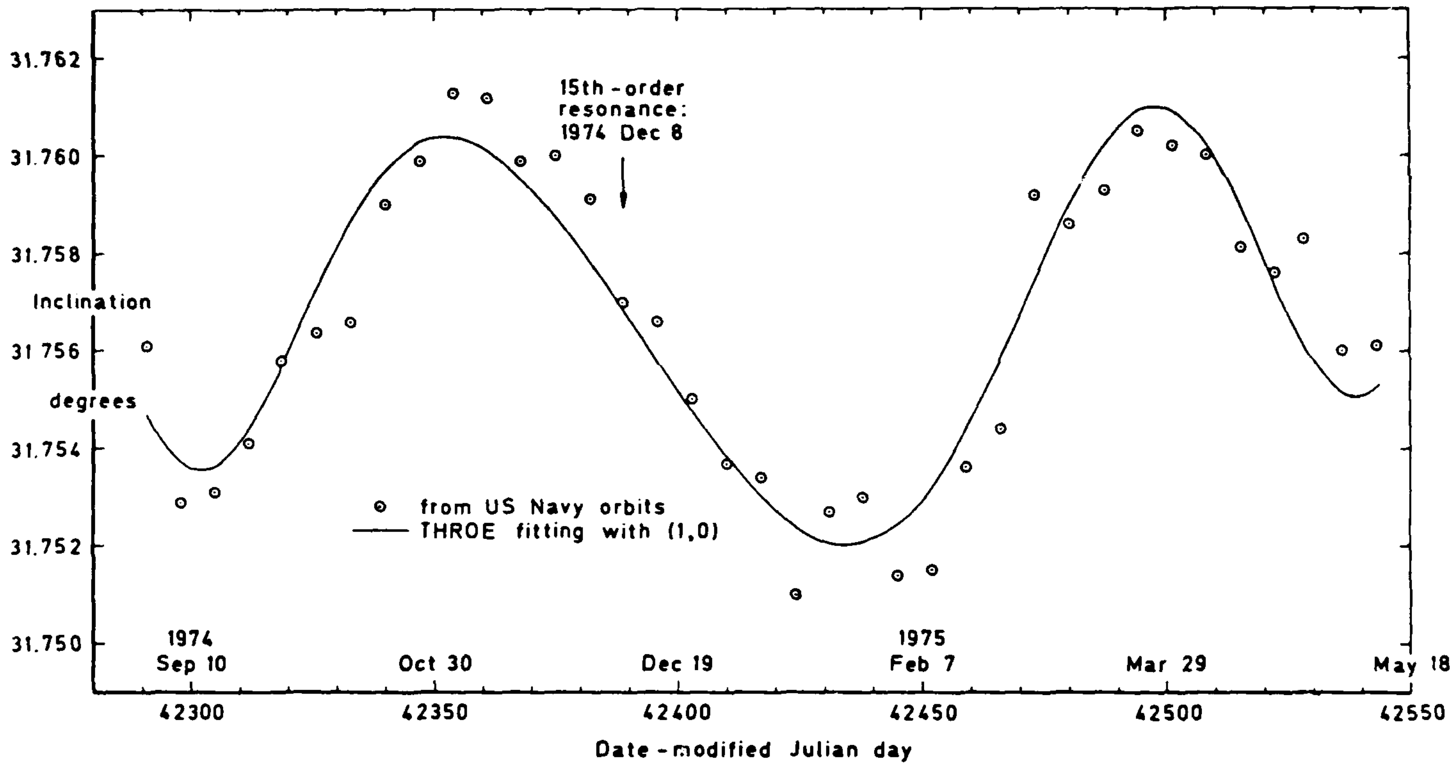

Fig 1 1965-09A: variation of inclination near 15th-order resonance

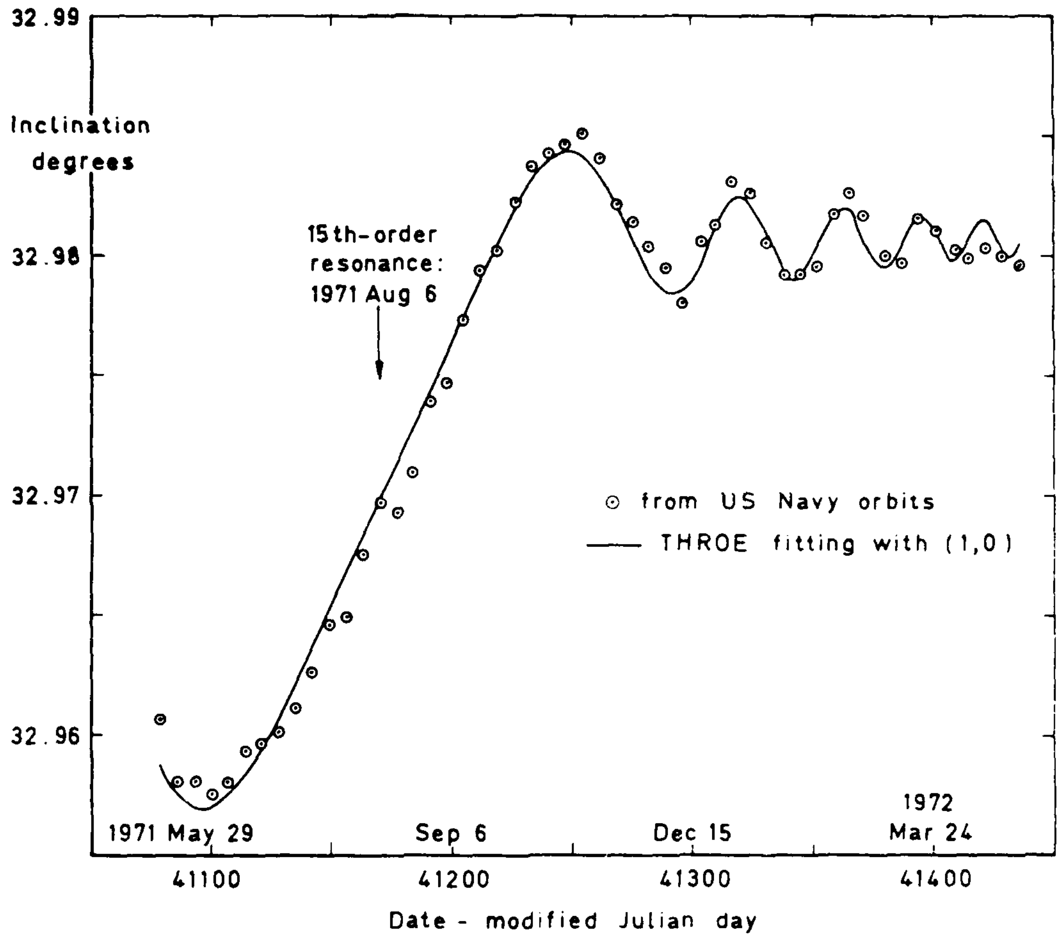

Fig 2 1969-68B. variation of inclination near 15th-order resonance

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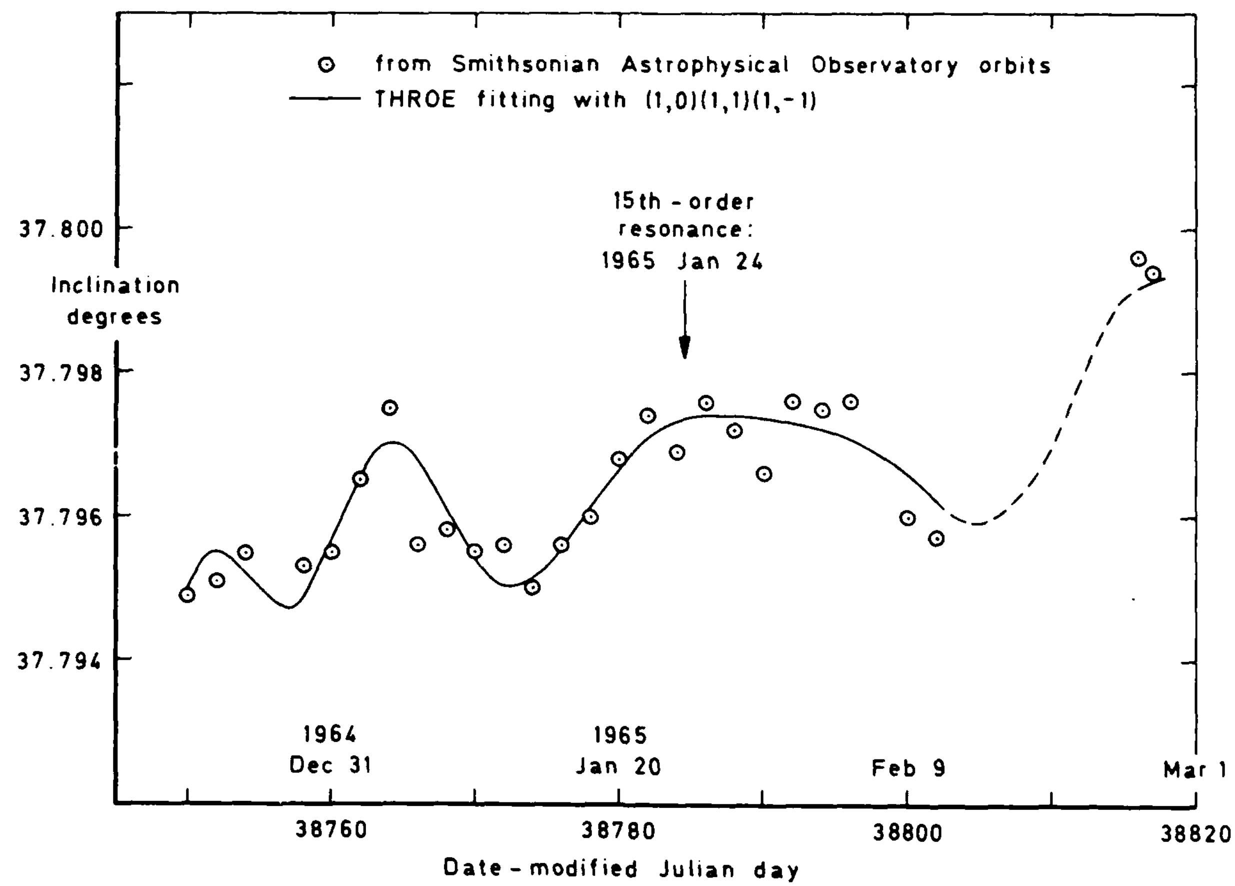

Fig 3 1964-84A: variation of inclination near 15th-order resonance

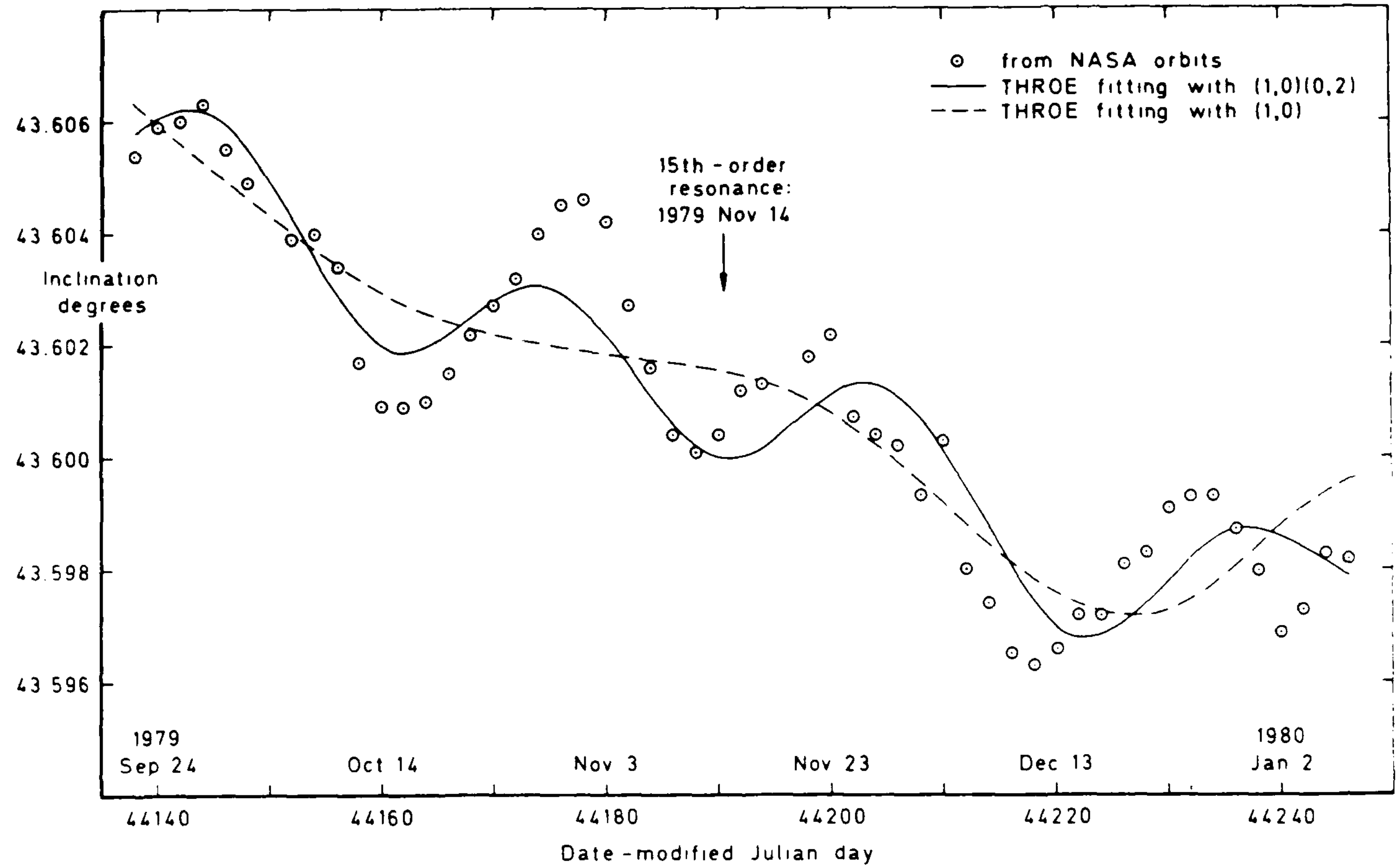

Fig 4 1979-82A: variation of inclination near 15th-order resonance

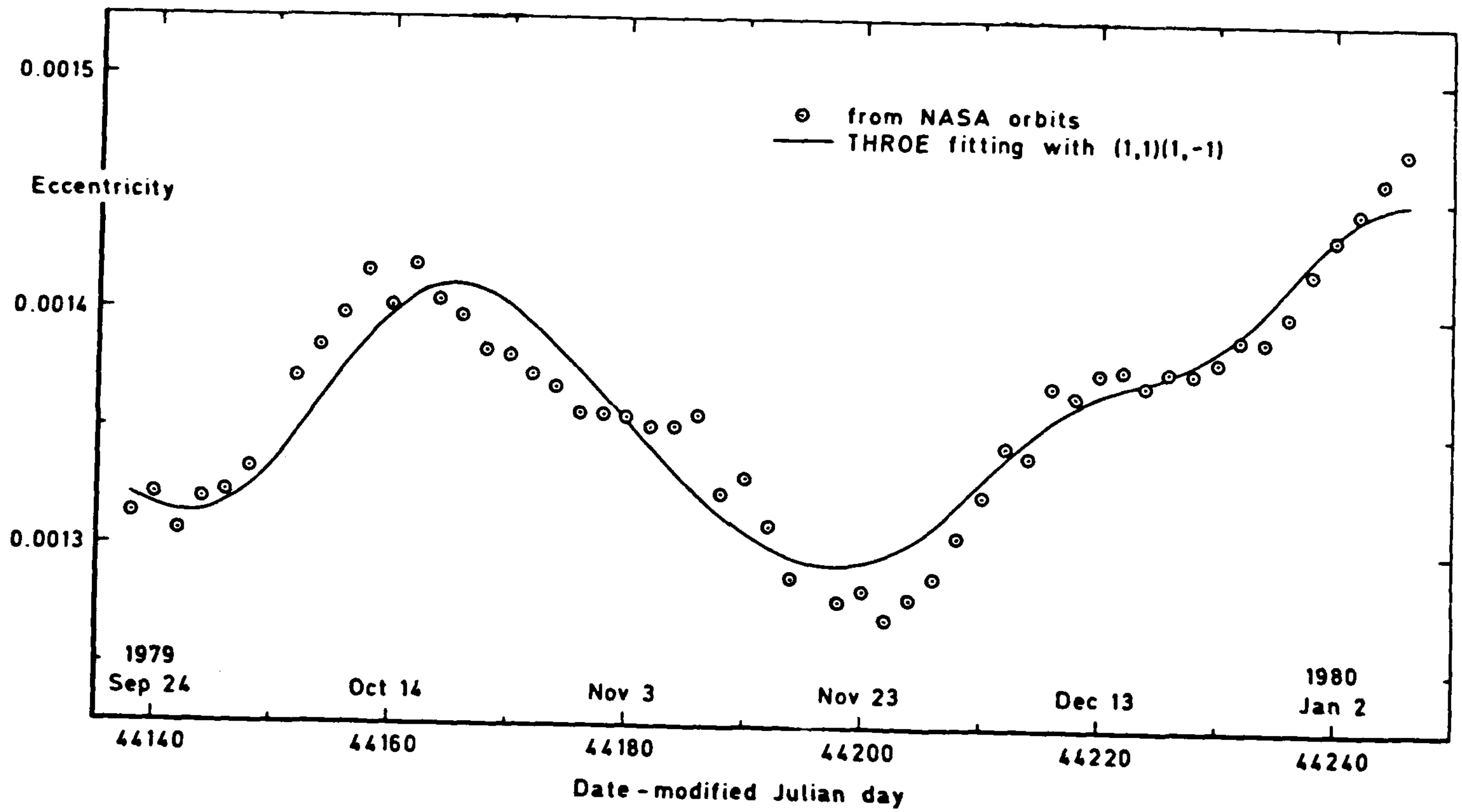

Fig 5 1979-82A: variation of eccentricity near 15th-order resonance

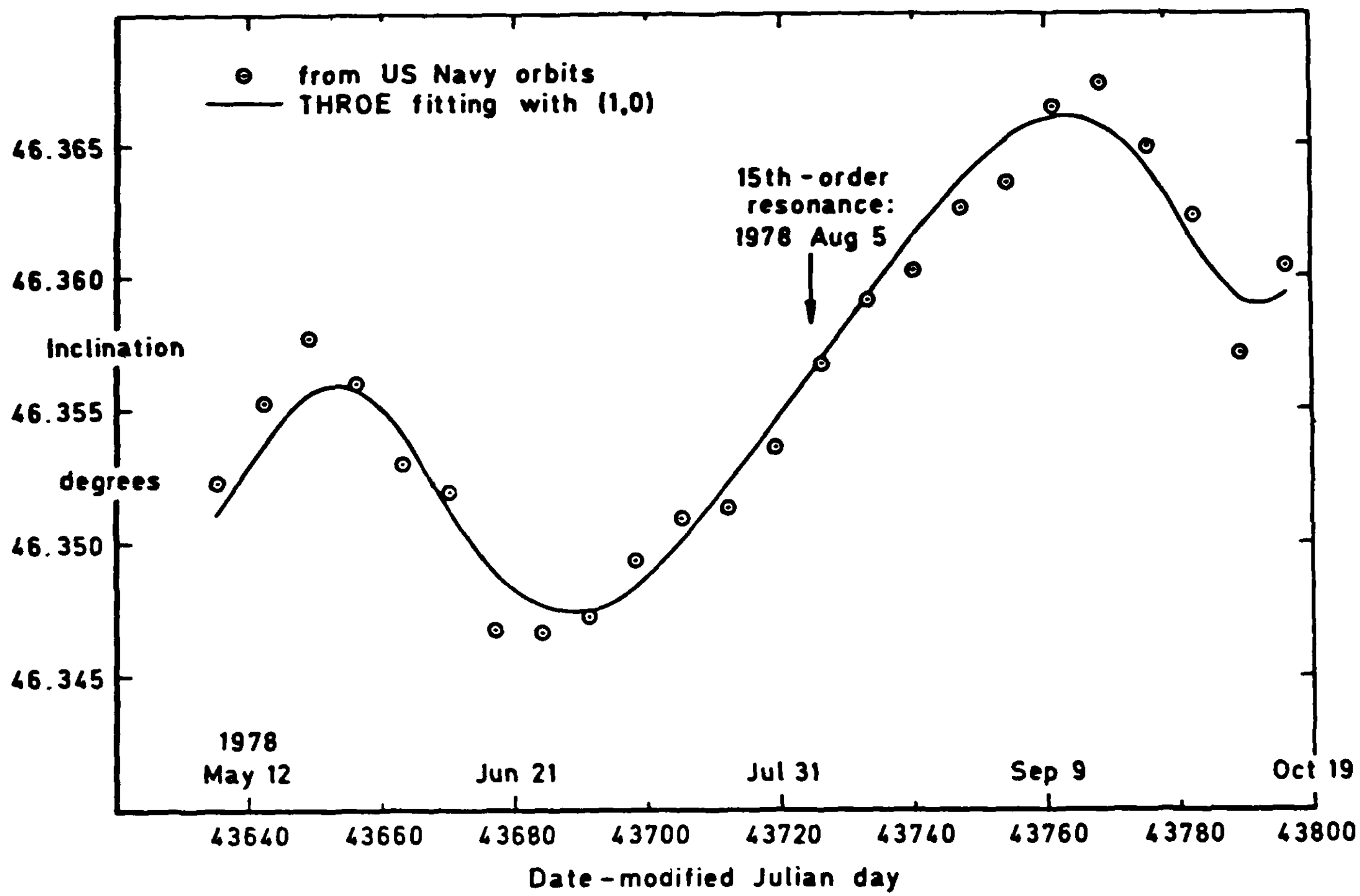

Fig 6 1971-30B: variation of inclination near 15th-order resonance

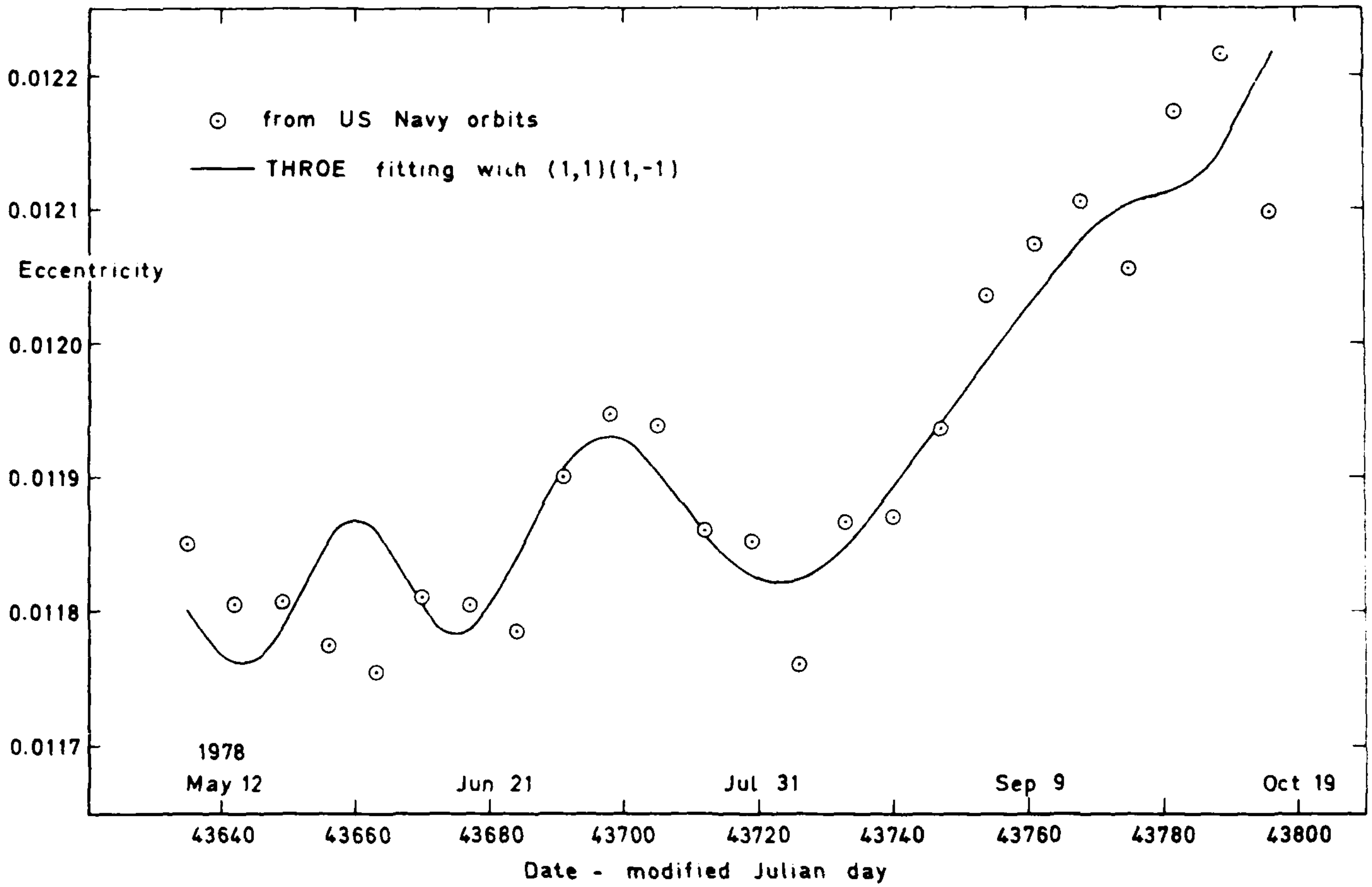

Fig 7 1971-30B: variation of eccentricity near 15th-order resonance

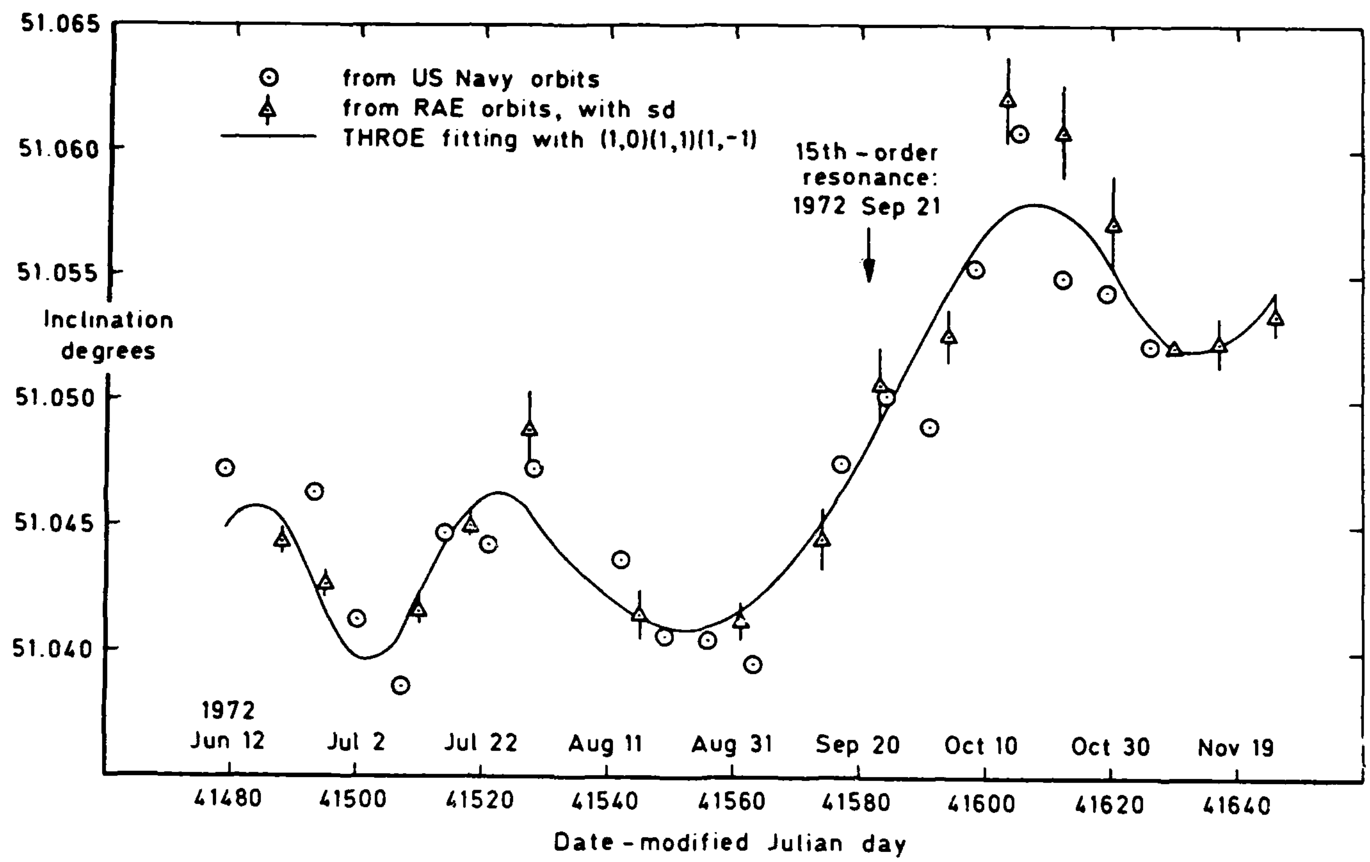

Fig 8 1871-58B: variation of inclination near 15th-order resonance

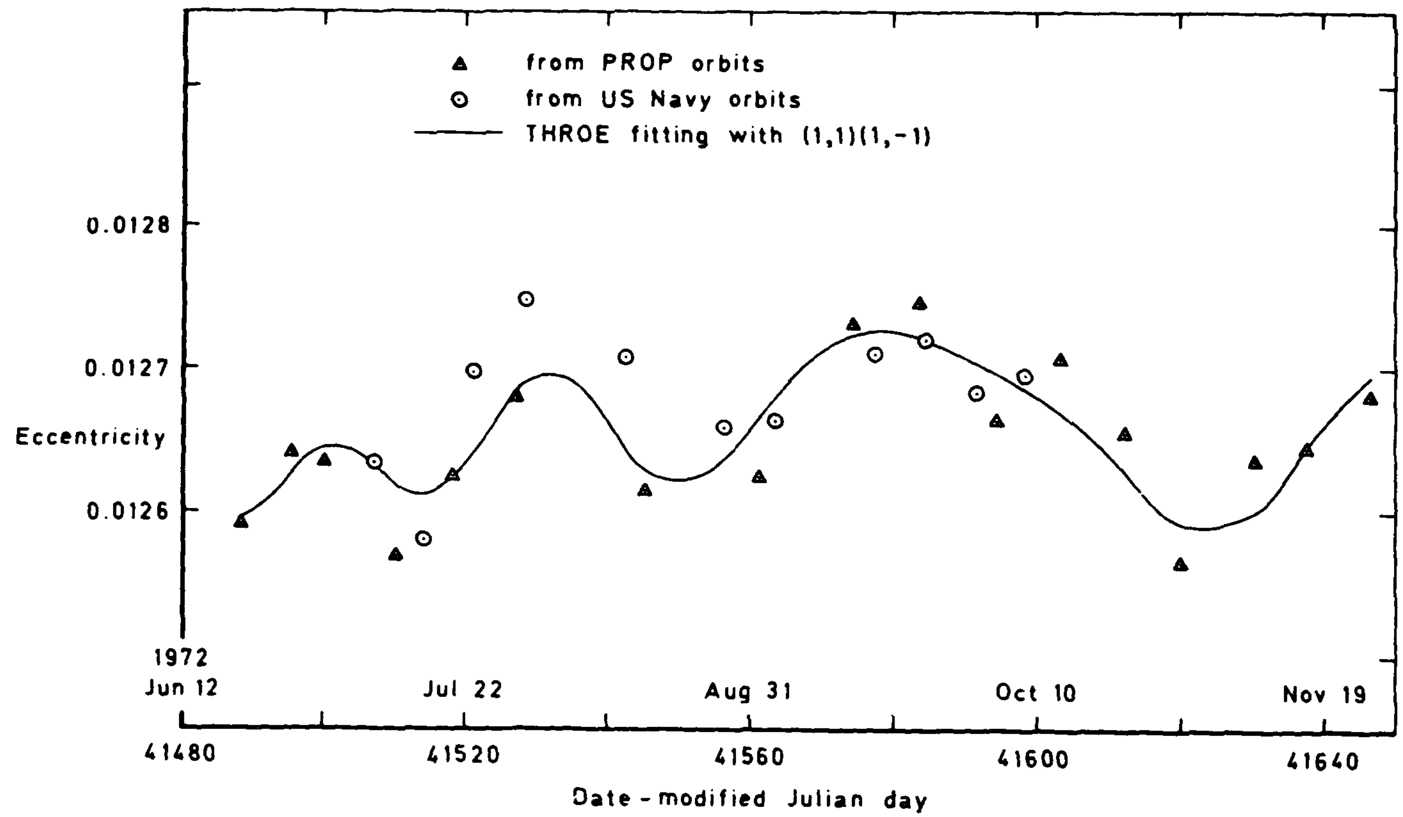

Fig 9 1971-58B: variation of eccentricity near 15th-order resonance

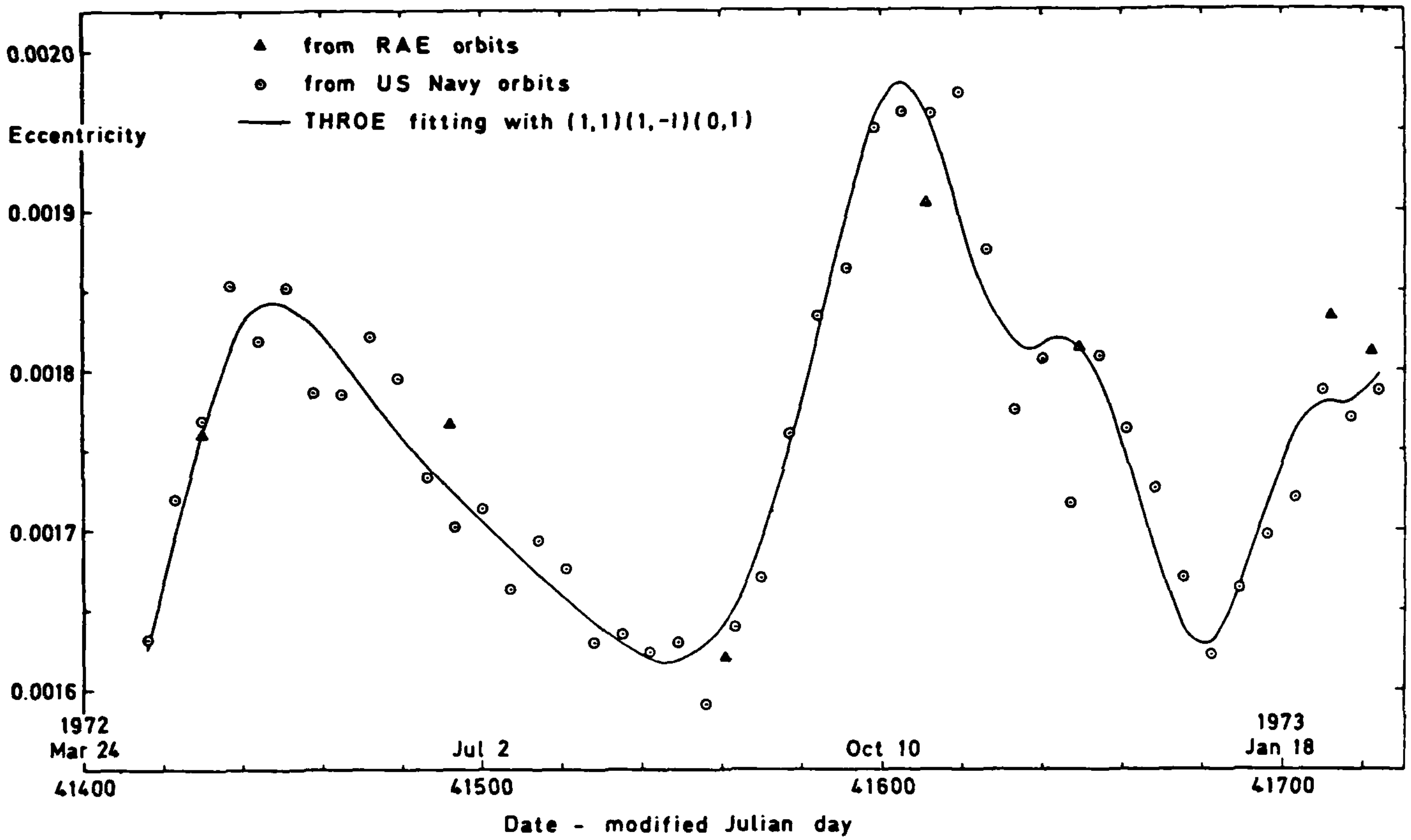

Fig 10 1965-53B: variation of eccentricity near 15th-order resonance

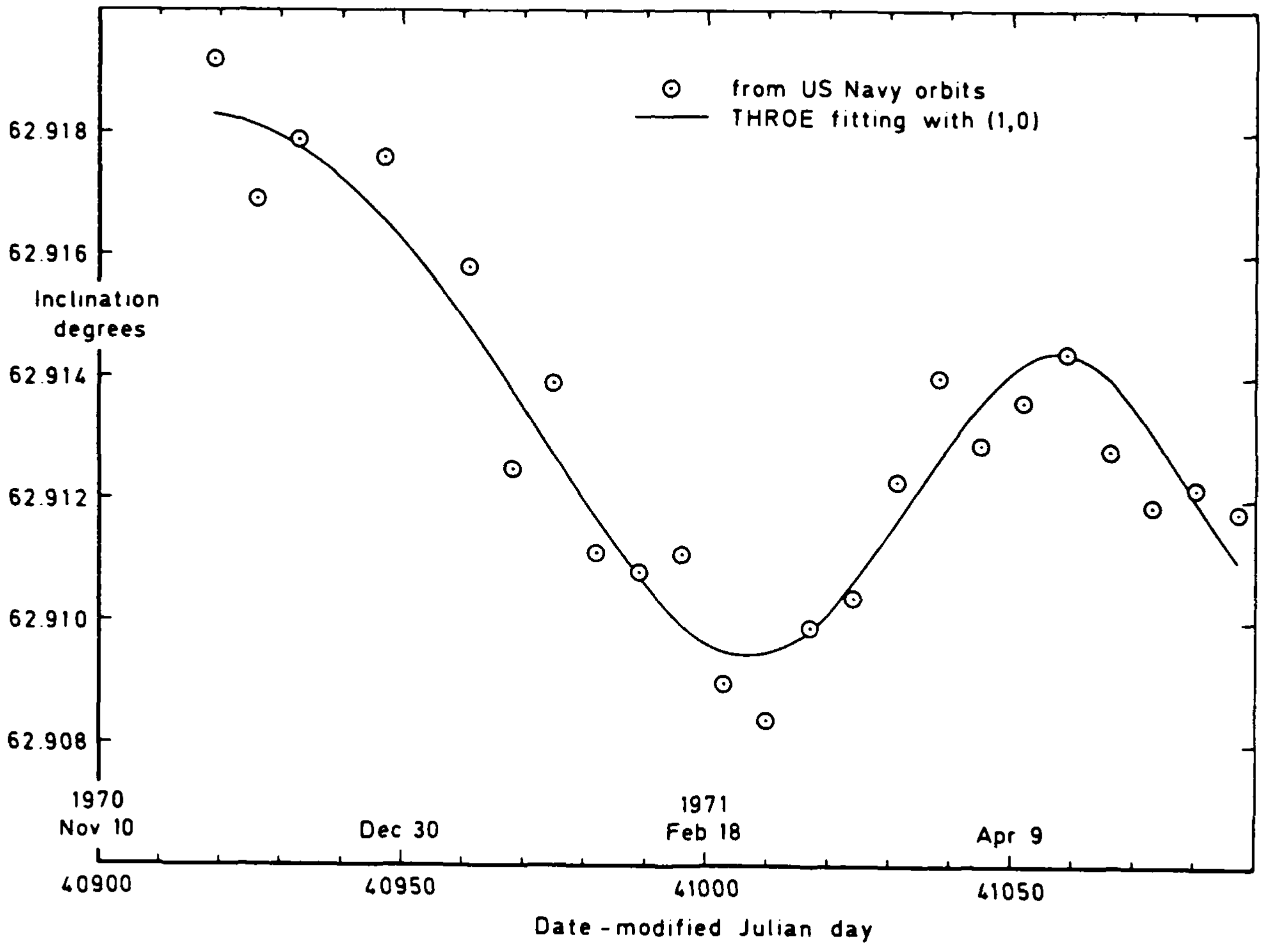

Fig 11 1970-87A: variation of inclination after 15th-order resonance

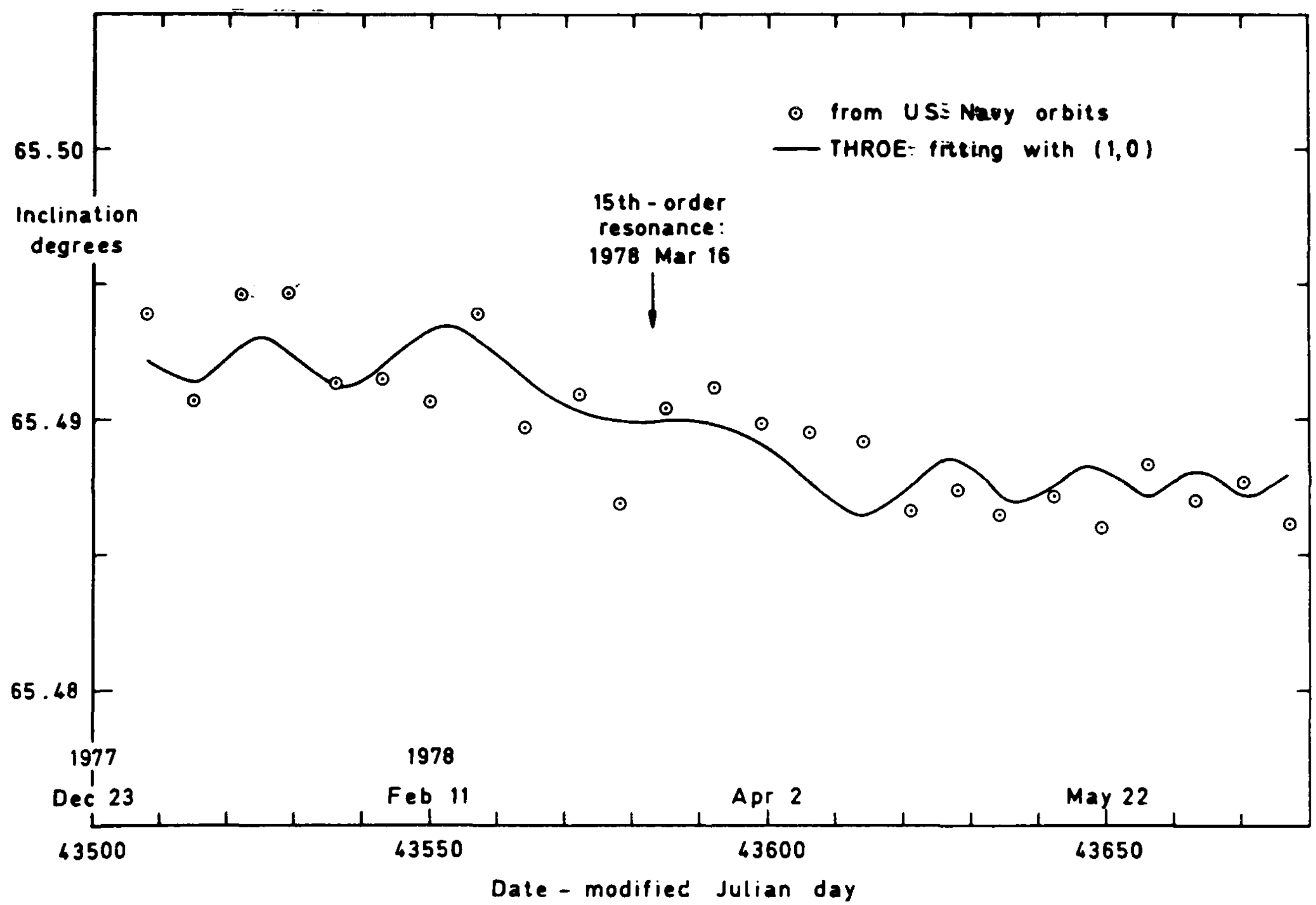

Fig 12 1977-12B: variation of inclination near 15th-order resonance

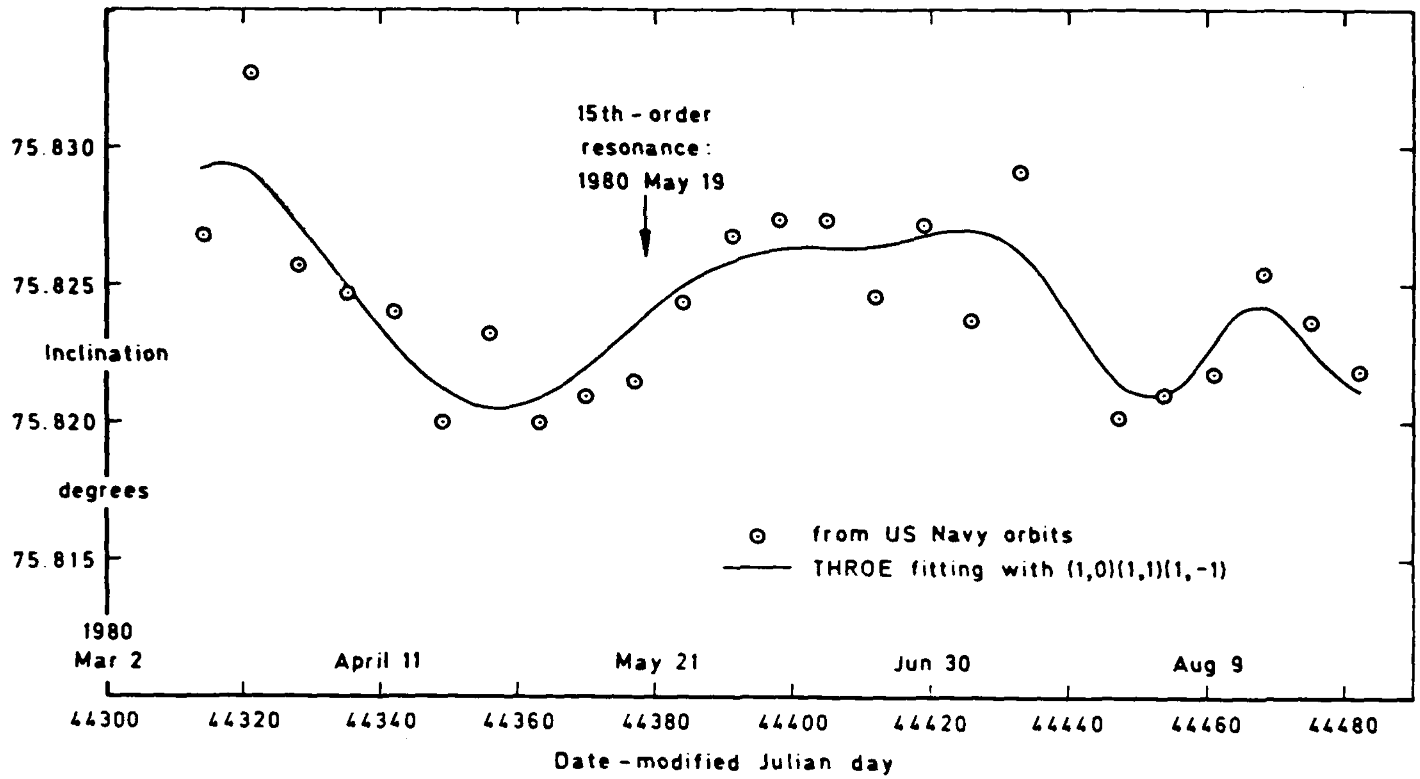

Fig 13 1977-95B: variation of inclination near 15th-order resonance

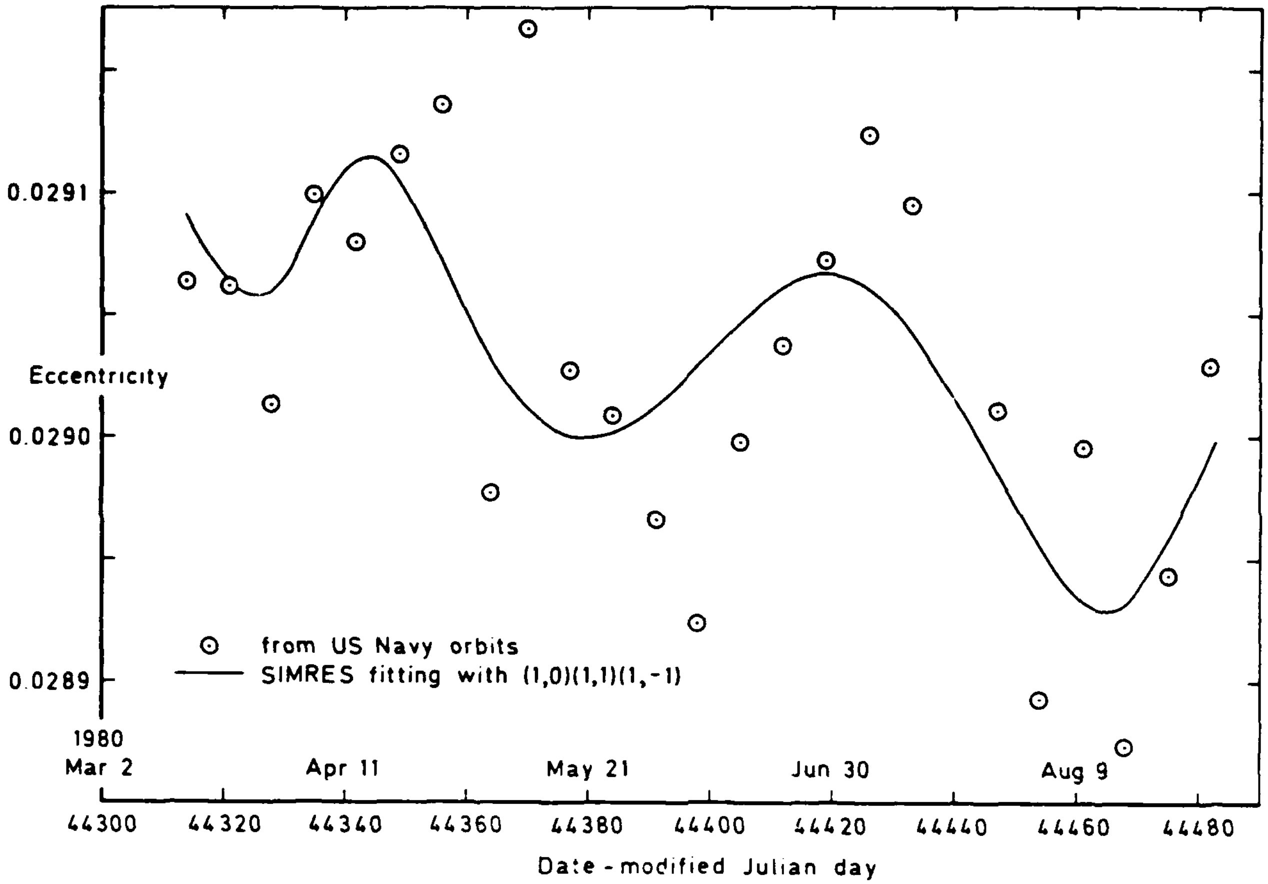

Fig 14 1977-95B: variation of eccentricity near 15th-order resonance

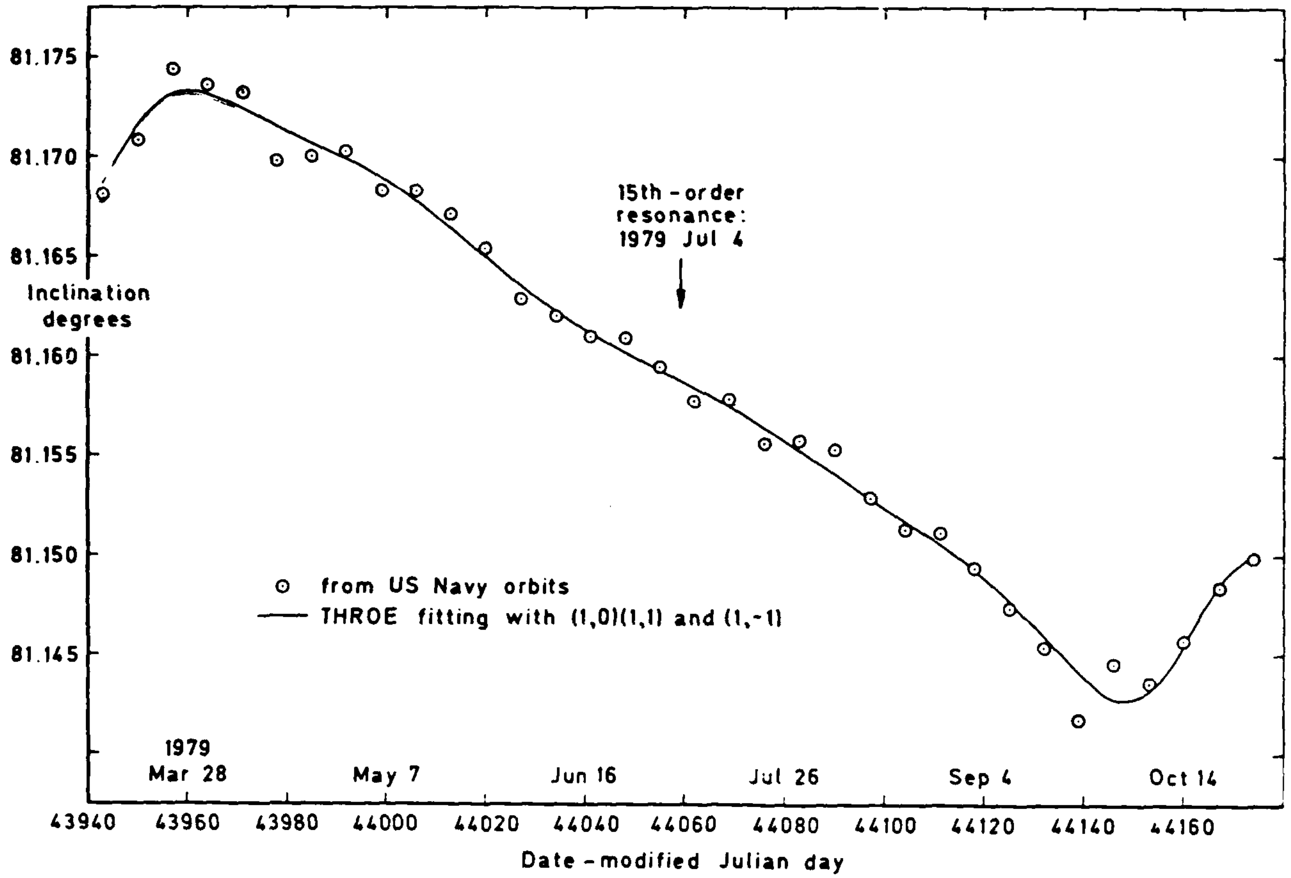

Fig 15 1970-19A: variation of inclination near 15th-order resonance

Fig 16

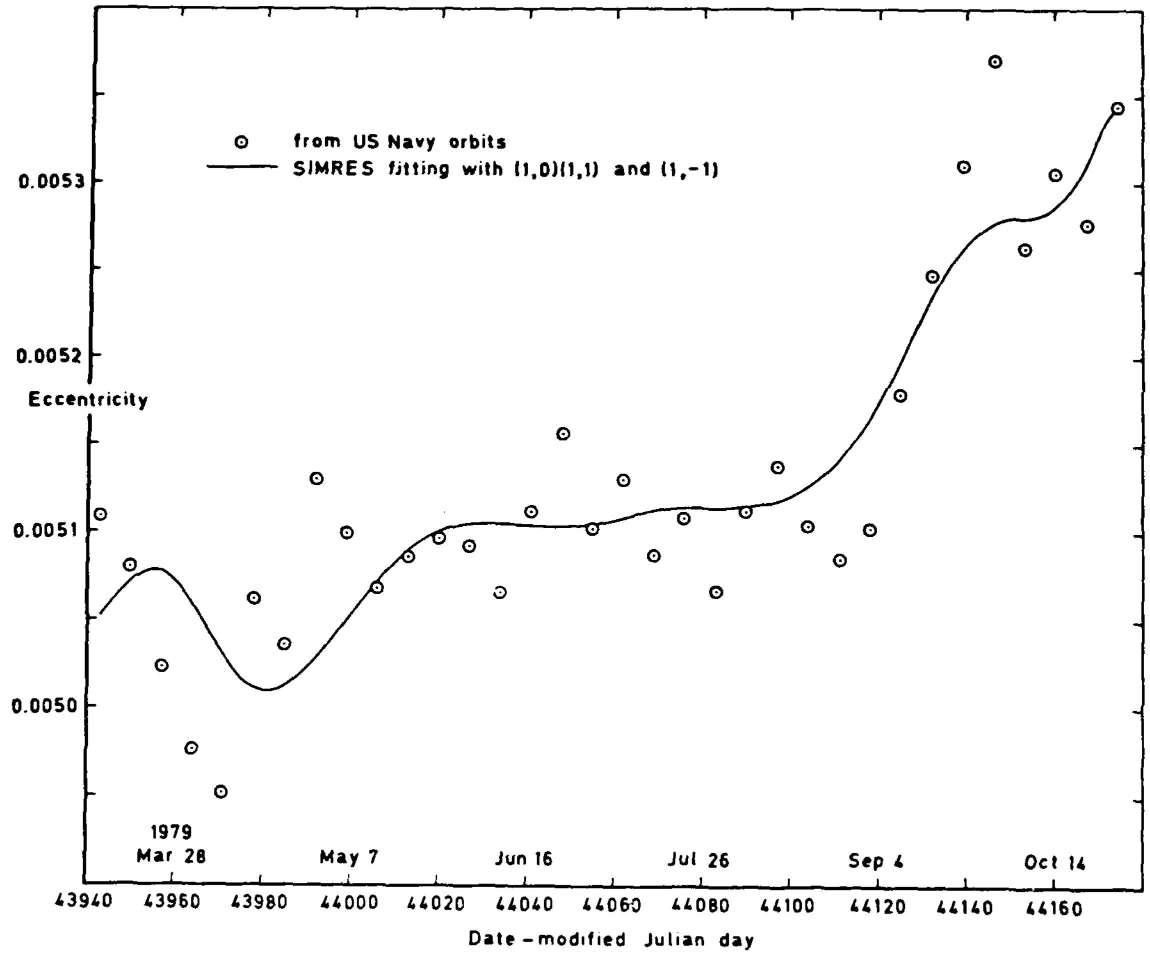

Fig 16 1970-19A: variation of eccentricity near 15th-order resonance

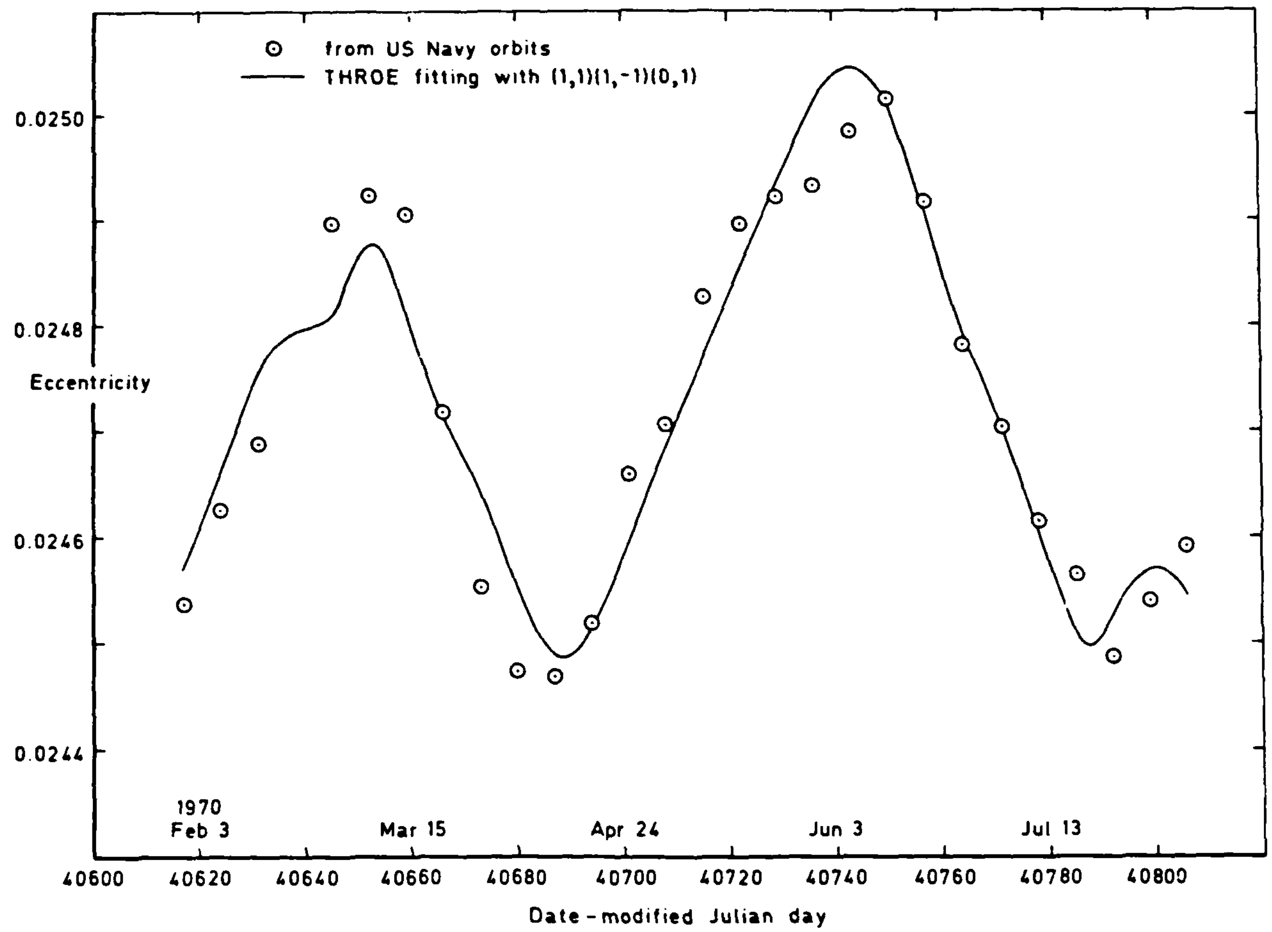

Fig 17 1967-73A: variation of eccentricity near 15th-order resonance

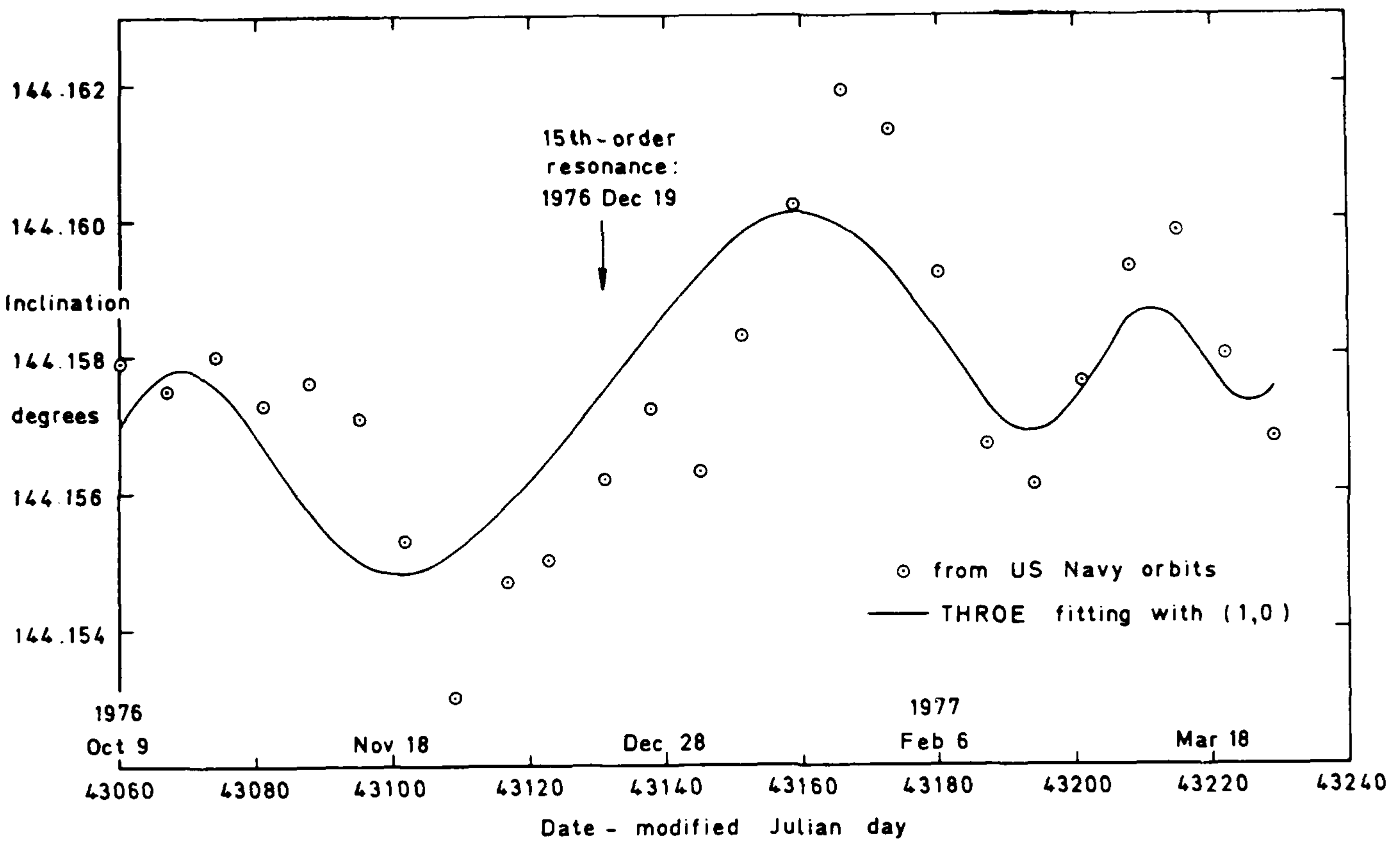

Fig 18 1966-63A: variation of inclination near 15th-order resonance

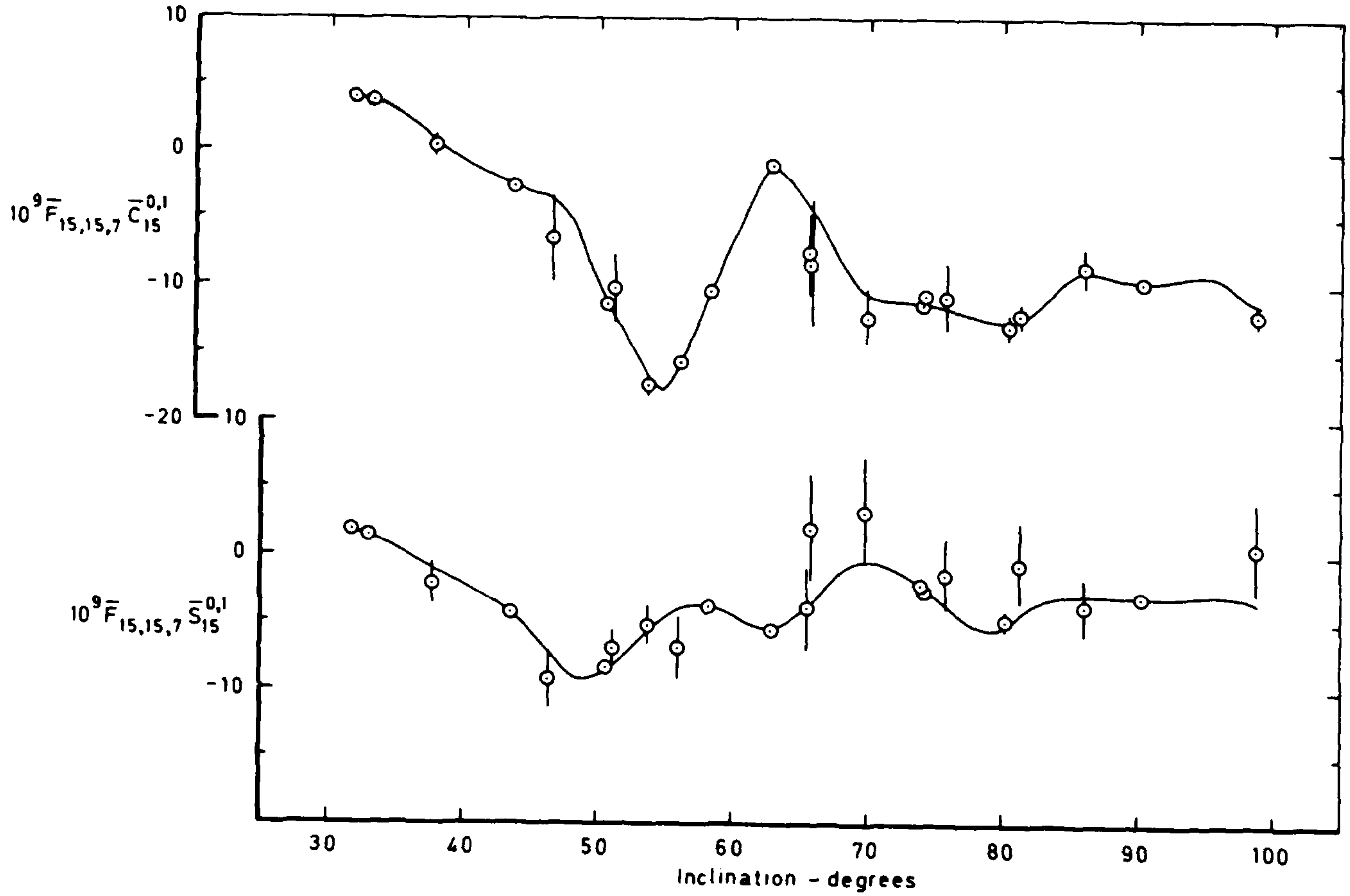

Fig 19 Values of $\bar{F}_{15,15,7}\bar{C}_{15}^{0,1}$ and $\bar{F}_{15,15,7}\bar{S}_{15}^{0,1}$ from Table 1 and the curves given by the 11-coefficient solution

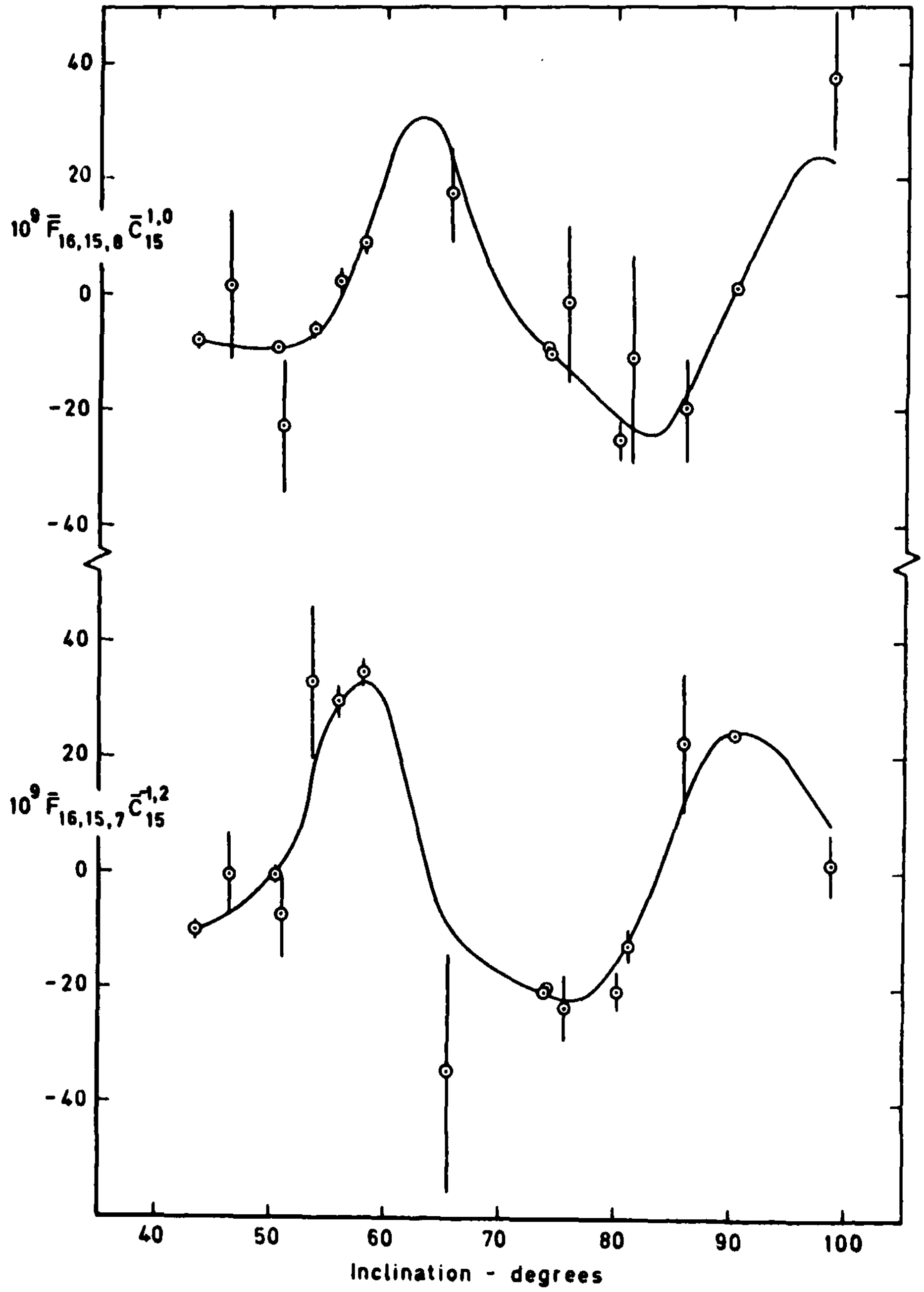

Fig 20 Values of $\bar{F}_{16,15,8}\bar{C}_{15}^{1,0}$ and $\bar{F}_{16,15,7}\bar{C}_{15}^{-1,2}$ from Table 2 and the curves given by the 10-coefficient solution

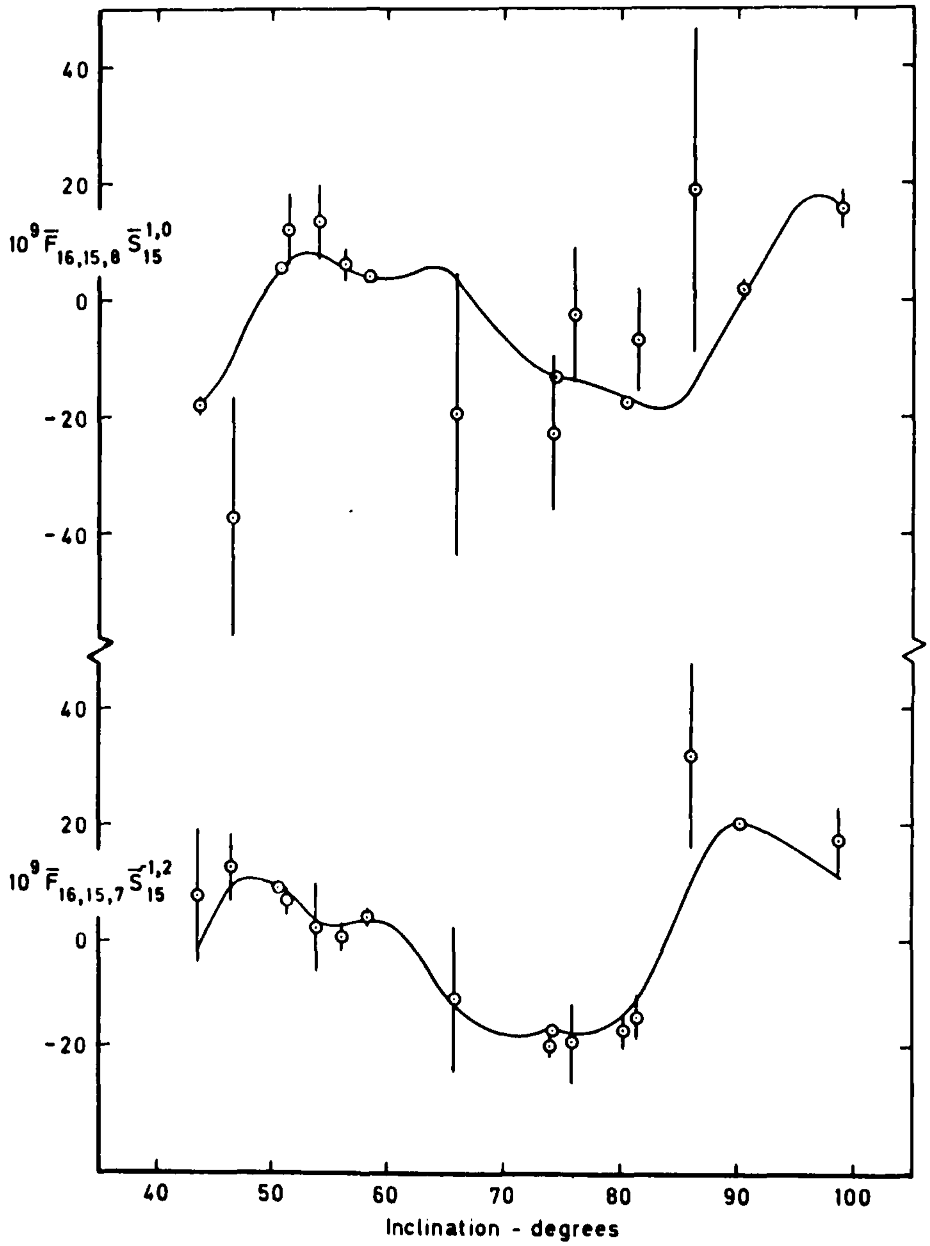

Fig 21 Values of $\bar{F}_{16,15,8}\bar{S}_{15}^{1,0}$ and $\bar{F}_{16,15,7}\bar{S}_{15}^{-1,2}$ from Table 2, and the curves given by the 10-coefficient solution

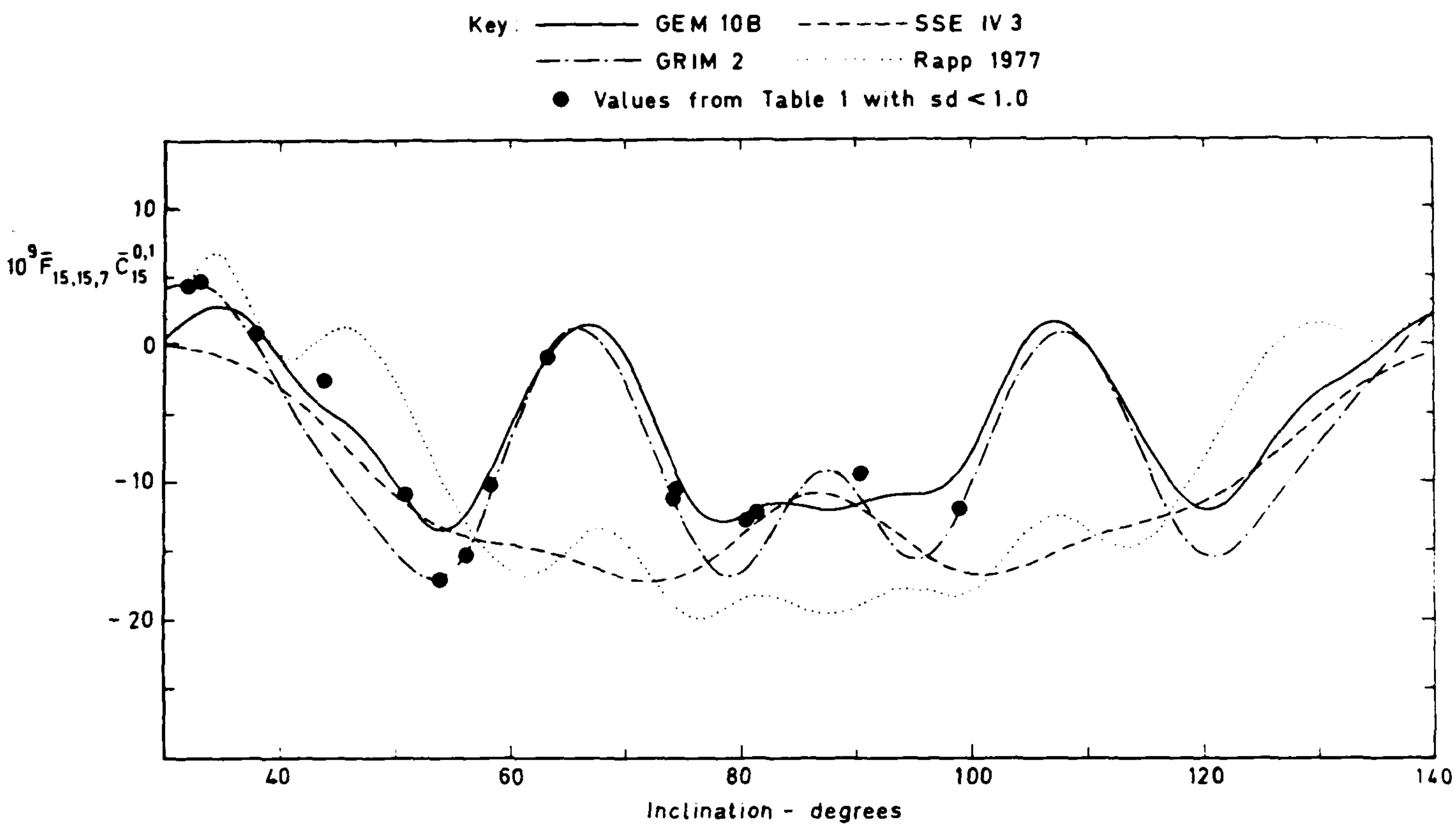

Fig 22 Variation of $\bar{F}_{15,15,7}\bar{C}_{15}^{0,1}$ with inclination as given by four Earth models, and values from Table 1

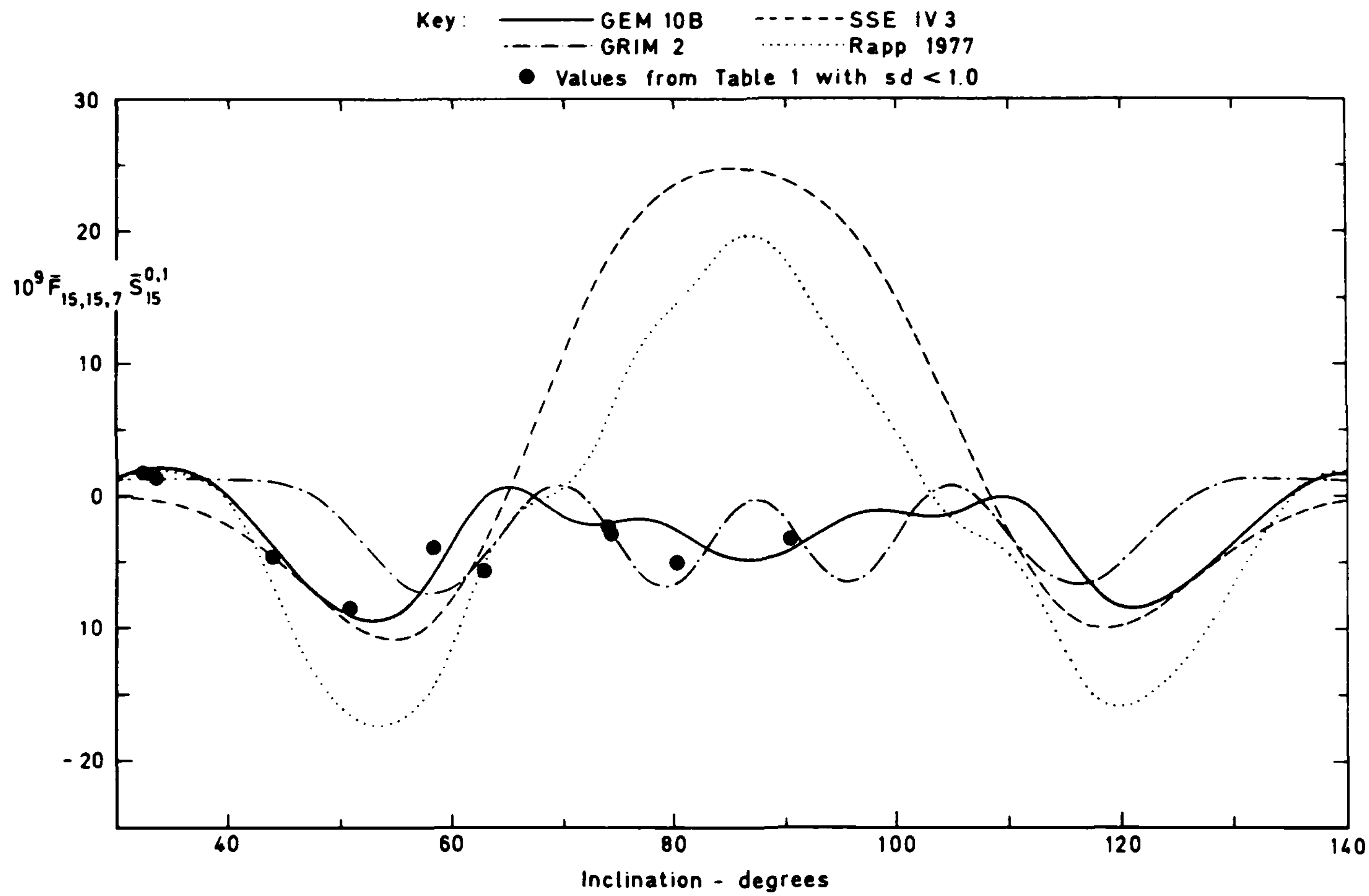

Fig 23 Variation of $\bar{F}_{15,15,7}\bar{S}_{15}^{0,1}$ with inclination as given by four Earth models, and values from Table 1

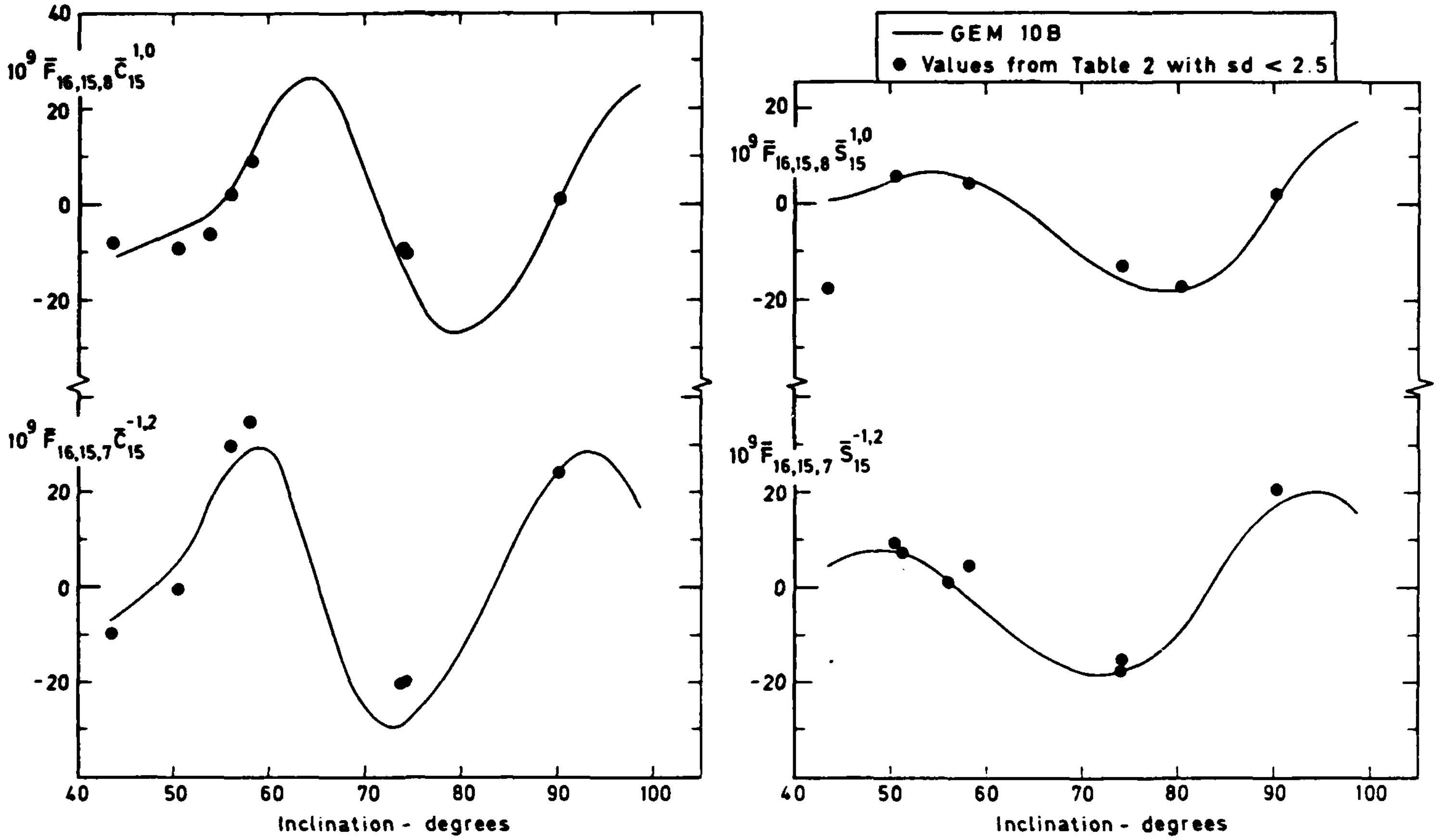

Fig 24 Variation with inclination of the four even-degree lumped harmonics as given by GEM 10B, and values from Table 2

REPORT DOCUMENTATION PAGE

Overall security classification of this page

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17. Abstract

Satellite orbits contracting under the influence of air drag experience 15th-order resonance when the track over the Earth repeats after 15 revolutions. If the orbital decay rate is slow enough, an orbit passing through the resonance is appreciably perturbed by the effects of 15th-order harmonics in the geopotential. We have used the observed perturbations in 23 resonant orbits, at various inclinations to the equator, to determine the harmonic coefficients of order 15 and degree 15, 16, 17, ... 35. Analysis of the changes in orbital inclination on the 23 orbits gives the harmonics of odd degree, while those of even degree are found from the changes in eccentricity on 16 of the orbits. The values derived are given in Tables 6 and 8. The coefficients of degrees 15, 16, 17, ... 23, should be more accurate than any previously obtained; their average sd is 1.4×10^{-9}, equivalent to 1 cm in geoid height.

Comparisons with comprehensive Earth models showed the Goddard Earth Model 10B to be the best, and a standard deviation of about 3×10^{-9} in the GEM 10B 15th-order coefficients is indicated.